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BESOV SPACES WITH NON-DOUBLING MEASURES

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ABSTRACT. Suppose that μ is a Radon measure on $\mathbb{R}^d$, which may be non-doubling. The only condition on μ is the growth condition, namely, there is a constant $C_0 > 0$ such that for all $x \in \operatorname{supp}(\mu)$ and $r > 0$,

$$\mu(B(x,r)) \le C_0 r^n,$$

where $0 < n \le d$. In this paper, the authors establish a theory of Besov spaces $\dot{B}^s_{pq}(\mu)$ for $1 \le p, q \le \infty$ and $|s| < \theta$, where $\theta > 0$ is a real number which depends on the non-doubling measure μ, C_0, n and d. The method used to define these spaces is new even for the classical case. As applications, the lifting properties of these spaces by using the Riesz potential operators and the dual spaces are obtained.

1. Introduction

Suppose that μ is a Radon measure on $\mathbb{R}^d$, which may be non-doubling. The only condition on μ is the growth condition, namely, there is a constant $C_0 > 0$ such that for all $x \in \operatorname{supp}(\mu)$ and $r > 0$,

$$\mu(B(x,r)) \le C_0 r^n, \tag{1.1}$$

where $0 < n \le d$.

Our goal in this paper is to develop a theory of Besov spaces associated to non-doubling measures on $\mathbb{R}^d$.

It is well known that the doubling property of the underlying measure is a basic condition in the classical Calderón-Zygmund theory of harmonic analysis. Recently more attention has been paid to non-doubling measures. It has been shown that many results of this theory still hold without assuming the doubling property; see [16, 17, 18, 19, 23, 24, 25, 29, 5, 6] for some results on Calderón-Zygmund operators, [15, 26, 27, 28] for some other results related to the spaces $BMO(\mu)$ and $H^1(\mu)$, and [7, 8, 20] for the vector-valued inequalities on the Calderón-Zygmund operators and weights.

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Let us first recall the definition of Besov spaces on $\mathbb{R}^d$. It is well known that the Littlewood-Paley theory gives a uniform treatment of function spaces on $\mathbb{R}^d$. Suppose that ψ is a Schwartz function satisfying the following conditions:

(i) $\operatorname{supp} \widehat{\psi} \subset \left\{\xi \in \mathbb{R}^d : 1/2 \le |\xi| \le 2\right\}$;

(ii) $\left|\widehat{\psi}(\xi)\right| \ge C > 0$ for all $3/5 \le |\xi| \le 5/3$.

The Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$ such that for $-\infty < s < \infty$ and $0 < p,\ q \le \infty$,

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{skq} \|\psi_k * f\|^q_{L^p(\mathbb{R}^d)} \right\}^{1/q} < \infty,$$

where $\psi_k(x) = 2^{kd}\psi(2^k x)$ for $x \in \mathbb{R}^d$ and $k \in \mathbb{Z}$, and $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$ is the space of Schwartz distributions modulo the space of all polynomials.

A key tool used to study these Besov spaces is the so-called Calderón reproducing formula which was first provided by Calderón in [1]. This formula says that for any given function ψ satisfying the above conditions (i) and (ii), there exists a function ϕ with the properties similar to ψ such that

$$f = \sum_{k=-\infty}^{\infty} \phi_k * \psi_k * f, \tag{1.2}$$

where the series converges in $L^2(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$; see [4].

Using this formula one can show that the Besov spaces $\dot{B}^s_{pq}(\mathbb{R}^d)$ are independent of the choice of ψ. Also using this formula one can establish the embedding theorems, the interpolation theorems, duality, atomic decomposition and the $T1$ theorems for the spaces $\dot{B}^s_{pq}(\mathbb{R}^d)$; see [4, 21, 31, 32] for more details.

By Coifman's ideas, David, Journé and Semmes in [3] provided the Littlewood-Paley theory for spaces of homogeneous type introduced by Coifman and Weiss in [2]. To be precise, if $\{S_k\}_{k=-\infty}^{\infty}$ is an approximation to the identity on a space of homogeneous type, then their kernels $\{S_k(x,y)\}_{k=1}^{\infty}$ satisfy certain size and regularity conditions; see [3] for the construction of such an approximation to the identity. It is worth pointing out that the doubling property plays an important role in this construction. Set $D_k = S_k - S_{k-1}$. Based on Coifman's ideas (see [3] for the details), at least formally, the identity operator I can be written as

$$\begin{aligned} I &= \sum_{k=-\infty}^{\infty} D_k \\ &= \left(\sum_{k=-\infty}^{\infty} D_k\right)\left(\sum_{j=-\infty}^{\infty} D_j\right) \\ &= \sum_{|k-j|\le N} D_k D_j + \sum_{|k-j|>N} D_k D_j \\ &= T_N + R_N. \end{aligned} \tag{1.3}$$

David, Journé and Semmes proved that if N is large enough, then R_N is bounded on $L^p(X), 1 < p < \infty$, with the operator norm less than 1. Therefore, if N is large enough and $D_k^N = \sum_{|j|\le N} D_{k+j}$ for $k \in \mathbb{Z}$, they obtained the following Calderón-type reproducing formulae:

$$f = \sum_{k=-\infty}^{\infty} T_N^{-1} D_k^N D_k(f) = \sum_{k=-\infty}^{\infty} D_k D_k^N T_N^{-1}(f), \tag{1.4}$$

where T_N^{-1} is the inverse of T_N and the series converge in $L^p(X)$, $1 < p < \infty$. Using this formula, they were able to obtain the Littlewood-Paley theory for the space $L^p(X)$: There exists a constant $C > 0$ such that for all $f \in L^p(X)$, $1 < p < \infty$,

$$C^{-1}\|f\|_{L^p(X)} \le \left\| \left\{ \sum_{k=-\infty}^{\infty} |D_k(f)|^2 \right\}^{1/2} \right\|_{L^p(X)} \le C\|f\|_{L^p(X)}.$$

In [12], via the Littlewood-Paley theory, Sawyer and the second author of this paper introduced the Besov space on spaces of homogeneous type. More precisely, they first introduced a space of test functions, $\mathcal{M}(X)$ (which is also called the space of smooth molecules in [9]), and approximations to the identity $\{S_k\}_{k=-\infty}^{\infty}$ whose kernels satisfy all size and regularity conditions (as mentioned above in Coifman's construction), and furthermore, the second difference smoothness condition. They then proved that if N is large enough, R_N is bounded on the space of test functions, $\mathcal{M}(X)$, with the operator norm less than 1. Using this fact, they obtained the Calderón reproducing formula. To be precise, let $\{S_k\}_{k=-\infty}^{\infty}$ be any approximation to the identity as in [12] and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}$. Then there exist families of operators $\{\widetilde{D}_k\}_{k=-\infty}^{\infty}$ and $\{\overline{D}_k\}_{k=-\infty}^{\infty}$ such that

$$f = \sum_{k=-\infty}^{\infty} \widetilde{D}_k D_k(f) = \sum_{k=-\infty}^{\infty} D_k \overline{D}_k(f), \tag{1.5}$$

where the series converge in the norms of the space $L^p(X)$, $1 < p < \infty$, the space $\mathcal{M}(X)$ and the dual space $(\mathcal{M}(X))'$, respectively.

Note that the formula in (1.5) is similar to that in (1.2), and the second difference smoothness condition plays a crucial role in establishing (1.5). Thus, the theory of Besov spaces on spaces of homogeneous type can be developed as in the case of $\mathbb{R}^d$. More precisely, the Besov space on a space of homogeneous type (X, ρ, μ), $\dot{B}^s_{p,q}(X)$ for $1 \le p,\ q \le \infty$ and $|s| < \theta$, where θ depends on the regularity of $S_k(x, y)$, is defined to be the set of all $f \in (\mathcal{M}(X))'$ such that

$$\|f\|_{\dot{B}^s_{p,q}(X)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{skq} \|D_k(f)\|^q_{L^p(X)} \right\}^{1/q} < \infty,$$

where $D_k(f)(x) = \langle D_k(x, \cdot), f(\cdot)\rangle$; see [12] (also [10]) for the details. Again, using formulae (1.5), one showed that Besov spaces are independent of the choice of approximations to the identity $\{S_k\}_{k=-\infty}^{\infty}$, and, moreover, all other properties such as embedding, interpolation, duality, atomic decomposition and the $T1$ theorem were obtained; see [12, 10, 11, 13, 14].

One of the main difficulties for developing a theory of Besov space with respect to some non-doubling measure μ which does not satisfy any regularity property, apart from the growth condition (1.1), is the construction of approximations to the identity. More recently, Tolsa constructed a "reasonable" approximation to the identity. To be precise, Tolsa in [27] constructed a sequence of integral operators

$\{S_k\}_{k=-\infty}^{\infty}$ given by kernels $\{S_k(x,y)\}_{k=-\infty}^{\infty}$ defined on $\mathbb{R}^d \times \mathbb{R}^d$. Moreover, these kernels satisfy some appropriate size and regularity conditions and

$$\int_{\mathbb{R}^d} S_k(x,y)d\mu(y) = 1$$

for all $x \in \operatorname{supp}(\mu)$ and $S_k(x,y) = S_k(y,x)$ for all $k \in \mathbb{Z}$. For each $k \in \mathbb{Z}$, set $D_k = S_k - S_{k-1}$, and then, again, based on Coifman's ideas mentioned above, and by use of the appropriate size and regularity conditions on $S_k(x,y)$, the Cotlar-Stein lemma (see [22]) and the Calderón-Zygmund theory associated to non-doubling measures, Tolsa proved that the Calderón-type reproducing formula in (1.4) still holds for non-doubling measures. Using this formula, Tolsa was able to establish a Littlewood-Paley theory associated to non-doubling measures. However, the size and regularity conditions of $S_k(x,y)$ constructed by Tolsa are not enough to obtain a Calderón reproducing formula similar to (1.5). A crucial observation of this paper is that if the norm $\|f\|_{\dot{B}^s_{pq}(\mu)}$ for all $L^2(\mu)$ functions f is defined by

$$\|f\|_{\dot{B}^s_{pq}(\mu)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{skq} \|D_k(f)\|^q_{L^p(\mu)} \right\}^{1/q} < \infty,$$

where $\{D_k\}_{k=-\infty}^{\infty}$ are the same as in Tolsa's Calderón-type reproducing formula, then R_N in (1.3) is bounded with respect to this norm and the operator norm is less than 1 if N is large enough. Hence, T_N^{-1} is bounded with respect to this norm. This observation leads us to introduce a new "space of test functions" defined by

$$\dot{\mathcal{B}}^s_{p,q}(\mu) = \left\{ f \in L^2(\mu) : \|f\|_{\dot{B}^s_{pq}(\mu)} < \infty \right\}.$$

We will prove that the Calderón-type reproducing formulae in (1.4) with Tolsa's approximations to the identity hold for the space of test functions, $\dot{\mathcal{B}}^s_{p,q}(\mu)$.

To show the Calderón-type reproducing formulae in (1.4) still hold on the "distribution (dual) space" $\left(\dot{\mathcal{B}}^s_{p,q}(\mu)\right)'$, as for spaces of homogeneous type mentioned above, the second difference smoothness condition of the approximation to the identity is needed. We will show that Tolsa's construction of approximations to identity does have this property (see Lemma 2.1 (f) below) and, hence, the Calderón-type reproducing formulae in (1.4) on the "distribution space" $\left(\dot{\mathcal{B}}^s_{p,q}(\mu)\right)'$ are obtained. As soon as the Calderón-type reproducing formulae on the distribution space are established, we can develop a theory of Besov spaces with non-doubling measures as in the cases of $\mathbb{R}^d$ and spaces of homogeneous type.

The plan of this paper is the following. In the next section, we will show that the approximation to the identity of Tolsa satisfies the second difference smoothness estimate. Also, in this section, we will prove that the operator T_N^{-1} is bounded with respect to the norm $\|\cdot\|_{\dot{B}^s_{pq}(\mu)}$. To this end, we first prove that R_N in (1.3) is bounded with respect to this norm with small operator norm; see Theorem 2.1 below. Some ideas of the proof of Theorem 2.1 are similar to the proof of the Cotlar-Stein lemma (see [22]). The main result of this section is the Calderón-type reproducing formulae on the distribution space $\left(\dot{\mathcal{B}}^s_{p,q}(\mu)\right)'$; see Theorem 2.2 below. In Section 3, we will introduce the Besov space $\dot{B}^s_{pq}(\mu)$ for $1 \le p,\ q \le \infty$ and $|s| < \theta$, where $\theta > 0$ is a real number which depends on the non-doubling measure μ, C_0, n and d; see Definition 2.5 below. Moreover, if μ is the d-dimensional Lebesgue measure on $\mathbb{R}^d$, then $\theta = 1/2$ (see Remark 2.1 below). In this section, we will also give some applications of these spaces. To be precise, we study the boundedness of the Riesz potential operators on these spaces, and using them, we prove that these spaces have lifting properties. Finally we consider their dual spaces in this section.

Throughout the paper, the letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as, C_0, do not change in different occurrences. The notation $A \sim B$ means that there is some constant $C > 0$ such that $C^{-1}A \le B \le CA$. For any index $q \in [1, \infty]$, we denote by q' the conjugate index, namely $1/q + 1/q' = 1$. We also denote $\mathbb{N} \cup \{0\}$ by $\mathbb{Z}_+$.

2. Calderón-type reproducing formulae

We use the same notation and definitions as in Tolsa [27] (also [28]), and for the reader's convenience, we recall some basic notation and definitions here; see [27, 28] for more details.

Throughout this paper, by a cube Q we mean a closed cube with sides parallel to the axes and centered at some point of $\operatorname{supp}(\mu)$. Also, ρQ is the cube concentric with Q whose side length is ρ times the side length of Q.

We will assume that the constant C_0 in (1.1) has been chosen big enough so that for all cubes $Q \subset \mathbb{R}^d$, we have

$$\mu(Q) \le C_0 \ell(Q)^n, \tag{2.1}$$

where $0 < n \le d$ and $\ell(Q)$ is the side length of the cube Q.

We first recall the definition of doubling cubes of Tolsa in [26, 27]. Given $\alpha > 1$ and $\beta > \alpha^n$, we say that the cube $Q \subset \mathbb{R}^d$ is (α, β)-doubling if $\mu(\alpha Q) \le \beta \mu(Q)$; see [26, 27] for the existence and some other basic properties of the doubling cubes.

For definiteness, if α and β are not specified, by a doubling cube we mean a $(2, 2^{d+1})$-doubling cube in what follows.

Given cubes $Q,\ R \subset \mathbb{R}^d$, we denote by z_Q the center of Q, and by Q_R the smallest cube concentric with Q containing Q and R.

Definition 2.1. Given two cubes $Q,\ R \subset \mathbb{R}^d$, we define

$$\delta(Q, R) = \max \left\{ \int_{Q_R \setminus Q} \frac{1}{|x - z_Q|^n} \, d\mu(x), \int_{R_Q \setminus R} \frac{1}{|x - z_R|^n} \, d\mu(x) \right\}.$$

We may treat points $x \in \operatorname{supp}(\mu)$ as if they were cubes (with $\ell(x) = 0$). So, for $x,\ y \in \operatorname{supp}(\mu)$ and some cube Q, the notations $\delta(x, Q)$ and $\delta(x, y)$ make sense; see [28, 27] for some useful properties of $\delta(\cdot, \cdot)$.

We now recall the definition of cubes of different generations in [27, 28]; see [27, 28] for more details.

Definition 2.2. We say that $x \in \operatorname{supp}(\mu)$ is a stopping point (or stopping cube) if $\delta(x, Q) < \infty$ for some cube $Q \ni x$ with $0 < \ell(Q) < \infty$. We say that $\mathbb{R}^d$ is an initial cube if $\delta(Q, \mathbb{R}^d) < \infty$ for some cube Q with $0 < \ell(Q) < \infty$. The cubes Q such that $0 < \ell(Q) < \infty$ are called transit cubes.

Let A be some big positive constant. In particular, we assume that A is much bigger than the constants ϵ_0, ϵ_1 and γ_0, which appear, respectively, in Lemma 3.1, Lemma 3.2 and Lemma 3.3 of [27]. Moreover, the constants A, ϵ_0, ϵ_1 and γ_0 depend only on C_0, n and d.

In what follows, for $\epsilon > 0$ and $a,\ b \in \mathbb{R}$, the notation $a = b \pm \epsilon$ does not mean any precise equality but the estimate $|a - b| \le \epsilon$.

Definition 2.3. Assume that $\mathbb{R}^d$ is not an initial cube. We fix some doubling cube $R_0 \subset \mathbb{R}^d$. This will be our "reference" cube. For each $j \in \mathbb{N}$, we let R_{-j} be some doubling cube concentric with R_0, containing R_0, and such that $\delta(R_0, R_{-j}) = jA \pm \epsilon_1$ (which exists because of Lemma 3.3 of [27]). If Q is a transit cube, we say that Q is a cube of generation $k \in \mathbb{Z}$ if it is a doubling cube, and for some cube R_{-j} containing Q we have

$$\delta(Q, R_{-j}) = (j + k)A \pm \epsilon_1.$$

If $Q \equiv \{x\}$ is a stopping cube, we say that Q is a cube of generation k if for some cube R_{-j} containing x we have

$$\delta(Q, R_{-j}) \le (j + k)A \pm \epsilon_1.$$

Definition 2.4. Assume that $\mathbb{R}^d$ is an initial cube. Then we choose $\mathbb{R}^d$ as our "reference": If Q is a transit cube, we say that Q is a cube of generation $k \ge 1$ if

$$\delta(Q, \mathbb{R}^d) = kA \pm \epsilon_1.$$

If $Q \equiv \{x\}$ is a stopping cube, we say that Q is a cube of generation $k \ge 1$ if

$$\delta(x, \mathbb{R}^d) \le kA \pm \epsilon_1.$$

Moreover, for all $k \le 0$ we say that $\mathbb{R}^d$ is a cube of generation k.

In what follows, for any $x \in \operatorname{supp}(\mu)$, we denote by $Q_{x,k}$ some fixed doubling cube centered at x of the kth generation.

It is easily seen that if A is big enough, then $\ell(Q_{x,k+1}) \le \ell(Q_{x,k})/10$. Thus, $\ell(Q_{x,k}) \to 0$ as $k \to \infty$. In fact, the following more precise result was established in [28] by Tolsa; see also [27].

Lemma 2.1. *If A is big enough, then there exists some $\eta > 0$ such that for $m \in \mathbb{N}$ and $2Q_{x,k} \cap 2Q_{y,k+m} \ne \emptyset$ with $x,\ y \in \operatorname{supp}(\mu)$, $\ell(Q_{y,k+m}) \le 2^{-\eta m}\ell(Q_{x,k})$.*

The constant η appearing in Lemma 2.1 actually represents some kind of regularity of the approximation to the identity of Tolsa in [27], which is very important in establishing the theory of Besov spaces on $\mathbb{R}^d$.

Definition 2.5. Let θ be half of the maximum η such that Lemma 2.1 holds.

Thus, θ depends only on μ, C_0, n and d. By the proof of Lemma 2.2 in [28], $\theta \in (0, \infty)$. Thus, the situation for spaces of non-homogeneous type is not the same as spaces of homogeneous type.

Remark 2.1. If μ is just the d-dimensional Lebesgue measure on $\mathbb{R}^d$, in this case, we can take R_0 in Definition 2.3 to be the unit cube centered at the original and R_{-j} for $j \in \mathbb{N}$ to be the cube centered at the original with side length 2^j. Let ω_{d-1} be the area of the unit ball in $\mathbb{R}^d$. Then $A = \omega_{d-1}$ in this case, and if Q is a cube of generation k, then $\ell(Q) \sim 2^{-k}$. Thus, in this case, η in Lemma 2.1 is equal to 1 and θ in Definition 2.5 is equal to $1/2$.

We now define

$$\sigma \equiv 100\epsilon_0 + 100\epsilon_1 + 12^{n+1}C_0.$$

We will also introduce two new constants $\alpha_1,\ \alpha_2 > 0$ such that

$$\epsilon_0,\ \epsilon_1,\ C_0 \ll \sigma \ll \alpha_1 \ll \alpha_2 \ll A.$$

Definition 2.6. Let $y \in \operatorname{supp}(\mu)$. If $Q_{y,k}$ is a transit cube, we denote by $Q^1_{y,k}$, $Q^2_{y,k}$, $\hat{Q}^2_{y,k}$ and $Q^3_{y,k}$ some doubling cubes centered at y containing $Q_{y,k}$ such that

$$\begin{aligned}
\delta(Q_{y,k}, Q^1_{y,k}) &= \alpha_1 \pm \epsilon_1,\\
\delta(Q_{y,k}, Q^2_{y,k}) &= \alpha_1 + \alpha_2 \pm \epsilon_1,\\
\delta(Q_{y,k}, \hat{Q}^2_{y,k}) &= \alpha_1 + \alpha_2 + \sigma \pm \epsilon_1,\\
\delta(Q_{y,k}, Q^3_{y,k}) &= \alpha_1 + \alpha_2 + 2\sigma \pm \epsilon_1.
\end{aligned}$$

Also, we denote by $\check{Q}^1_{y,k}$ and $\check{\check{Q}}^1_{y,k}$ some doubling cubes centered at y and contained in $Q_{y,k-1}$ satisfying

$$\begin{aligned}
\delta(\check{Q}^1_{y,k}, Q_{y,k-1}) &= A - \alpha_1 + \sigma \pm \epsilon_1,\\
\delta(\check{\check{Q}}^1_{y,k}, Q_{y,k-1}) &= A - \alpha_1 + 2\sigma \pm \epsilon_1.
\end{aligned}$$

If any of the cubes $\check{\check{Q}}^1_{y,k}$ and $\check{Q}^1_{y,k}$ does not exist because $\delta(y, Q_{y,k})$ is not big enough, then we let it be the point y.

If $Q_{y,k} = \mathbb{R}^d$, we set $Q^1_{y,k} = Q^2_{y,k} = \hat{Q}^2_{y,k} = Q^3_{y,k} = \check{Q}^1_{y,k} = \check{\check{Q}}^1_{y,k} = \mathbb{R}^d$.

If $Q_{y,k} \equiv \{y\}$ is a stopping cube and $Q_{y,k-1} \equiv \{y\}$ is also a stopping cube, we set $Q^1_{y,k} = Q^2_{y,k} = \hat{Q}^2_{y,k} = Q^3_{y,k} = \{y\}$. If $Q_{y,k} \equiv \{y\}$ is a stopping cube but $Q_{y,k-1}$ is not, then we choose $Q^1_{y,k}$, $Q^2_{y,k}$, $\hat{Q}^2_{y,k}$ and $Q^3_{y,k}$ so that they are contained in $Q_{y,k-1}$, centered at y and

$$\begin{aligned}
\delta(Q^1_{y,k}, Q_{y,k-1}) &= A - \alpha_1 \pm \epsilon_1,\\
\delta(Q^2_{y,k}, Q_{y,k-1}) &= A - \alpha_1 - \alpha_2 \pm \epsilon_1,\\
\delta(\hat{Q}^2_{y,k}, Q_{y,k-1}) &= A - \alpha_1 - \alpha_2 - \sigma \pm \epsilon_1,\\
\delta(Q^3_{y,k}, Q_{y,k-1}) &= A - \alpha_1 - \alpha_2 - 2\sigma \pm \epsilon_1.
\end{aligned}$$

If any of these cubes does not exist because $\delta(y, Q_{y,k-1})$ is not big enough, we let this cube be the point y.

The following lemma is established in [28]; see also [27].

Lemma 2.2. *Let $y \in \operatorname{supp}(\mu)$. If we choose the constants α_1, α_2 and A big enough, we have*

$$Q_{y,k} \subset \check{\check{Q}}^1_{y,k} \subset \check{Q}^1_{y,k} \subset Q^1_{y,k} \subset Q^2_{y,k} \subset \hat{Q}^2_{y,k} \subset Q^3_{y,k} \subset Q_{y,k-1}.$$

For a fixed k, cubes of the kth generation may have very different sizes for different y's. The same happens for the cubes $Q^1_{y,k}$ and $Q^2_{y,k}$. Nevertheless, in [28] (see also [27]), it has been shown that we still have some kind of regularity.

Lemma 2.3. *Given x, $y \in \operatorname{supp}(\mu)$, let Q_x and Q_y be the cubes centered at x and y respectively, and assume that $Q_x \cap Q_y \neq \emptyset$ and that there exists some cube R_0 containing $Q_x \cup Q_y$ with $|\delta(Q_x, R_0) - \delta(Q_y, R_0)| \le 10\epsilon_1$. If R_y is some cube centered at y containing Q_y with $\delta(Q_y, R_y) \ge \sigma - 10\epsilon_1$, then $Q_x \subset R_y$. As a consequence, we have if $Q_{x,k} \cap Q_{y,k} \neq \emptyset$, then $Q_{x,k} \subset Q_{y,k-1}$.*

To recall the construction of the approximation to the identity of Tolsa in [27], we first recall the following auxiliary functions $\psi_{y,k}$ and $\varphi_{x,k}$.

Definition 2.7. For any $y \in \operatorname{supp}(\mu)$, the function $\psi_{y,k}$ is a function on $\mathbb{R}^d$ such that

$$
\begin{aligned}
&(a) \qquad 0 \le \psi_{y,k}(x) \le \min\left(\frac{4}{\ell(Q^1_{y,k})^n},\ \frac{1}{|y-x|^n}\right),\\
&(b) \qquad \psi_{y,k}(x) = \frac{1}{|x-y|^n} \quad if \quad x \in \hat{Q}^2_{y,k} \setminus Q^1_{y,k},\\
&(c) \qquad \operatorname{supp}(\psi_{y,k}) \subset Q^3_{y,k},\\
&(d) \qquad |\psi'_{y,k}(x)| \le C \min\left(\frac{1}{\ell(Q^1_{y,k})^{n+1}},\ \frac{1}{|y-x|^{n+1}}\right),
\end{aligned}
$$

where $C > 0$ is large enough.

Choosing $\alpha_2 > 0$ big enough, for all $y \in \operatorname{supp}(\mu)$, we define $\varphi_{y,k}(x) = \alpha_2^{-1}\psi_{y,k}(x)$.

Using $\{\varphi_{y,k}\}$, we can recall the definition of the approximation to the identity $\{S_k\}_{k\in\mathbb{Z}}$ in [27].

Definition 2.8. Let $f \in L^1_{\mathrm{loc}}(\mu)$ and $x \in \operatorname{supp}(\mu)$. If $Q_{x,k} \ne \mathbb{R}^d$, then we set

$$
\widetilde{S}_k f(x) = \int_{\mathbb{R}^d} \varphi_{y,k}(x) f(y)\, d\mu(y) + \max\left(0, \frac{1}{4} - \int_{\mathbb{R}^d} \varphi_{y,k}(x)\, d\mu(y)\right) f(x).
$$

Assume that $Q_{x,k} \ne \mathbb{R}^d$ for some $x \in \operatorname{supp}(\mu)$. Let M_k be the operator of multiplication by $M_k(x) \equiv 1/\widetilde{S}_k 1(x)$ and let W_k be the operator of multiplication by

$$
W_k(x) \equiv 1/\widetilde{S}^*_k(1/\widetilde{S}_k 1)(x).
$$

We set $S_k \equiv M_k \widetilde{S}_k W_k \widetilde{S}^*_k M_k$. If $Q_{x,k} = \mathbb{R}^d$ for some $x \in \operatorname{supp}(\mu)$, then we set $S_k \equiv 0$.

Observe that, formally, $\widetilde{S}_k$ is an integral operator with the following positive kernel:

$$
\widetilde{S}_k(x,y) = \varphi_{y,k}(x) + \max\left(0, \frac{1}{4} - \int_{\mathbb{R}^d} \varphi_{y,k}(x)\, d\mu(y)\right) dz_x(y),
$$

where δ_x is the Dirac delta at x, and that if $Q_{x,k}$ and $Q_{y,k}$ are transit cubes, then S_k is also an integral operator with the following positive kernel:

$$
S_k(x,y) = \int_{\mathbb{R}^d} M_k(x)\widetilde{S}_k(x,z) W_k(z) \widetilde{S}_k(y,z) M_k(y)\, d\mu(z). \tag{2.2}
$$

We now recall some basic properties of the kernels $\{S_k(x,y)\}_{k\in\mathbb{Z}}$ in (2.2), moreover, we will verify that they also have the regularity of the second difference.

Lemma 2.4. *There exist a sequence of operators of $\{S_k\}_{k=-\infty}^{\infty}$ with kernels $S_k(x,y)$ defined on $\mathbb{R}^d \times \mathbb{R}^d$. For each $k \in \mathbb{Z}$ the following properties hold:*

(a) $S_k(x,y) = S_k(y,x)$.
(b) $\int_{\mathbb{R}^d} S_k(x,y)\, d\mu(y) = 1$ *for* $x \in \operatorname{supp}(\mu)$.
(c) *If $Q_{x,k}$ is a transit cube, then* $\operatorname{supp}(S_k(x,\cdot)) \subset Q_{x,k-1}$.
(d) *If $Q_{x,k}$ and $Q_{y,k}$ are transit cubes, then*

$$
0 \le S_k(x,y) \le \frac{C}{(\ell(Q_{x,k}) + \ell(Q_{y,k}) + |x-y|)^n}.
$$

(e) *If* $Q_{x,k}$, $Q_{x',k}$, $Q_{y,k}$ *are transit cubes, and* x, $x' \in Q_{x_0,k}$ *for some* $x_0 \in \mathrm{supp}\,(\mu)$, *then*

$$|S_k(x,y) - S_k(x',y)| \le C\frac{|x-x'|}{\ell(Q_{x_0,k})}\frac{1}{(\ell(Q_{x,k}) + \ell(Q_{y,k}) + |x-y|)^n}.$$

(f) *If* $Q_{x,k}$, $Q_{x',k}$, $Q_{y,k}$ *and* $Q_{y',k}$ *are transit cubes,* x, $x' \in Q_{x_0,k}$ *and* y, $y' \in Q_{y_0,k}$ *for some* x_0, $y_0 \in \mathrm{supp}\,(\mu)$, *then*

$$\begin{aligned}(2.3)\qquad &|[S_k(x,y) - S_k(x',y)] - [S_k(x,y') - S_k(x',y')]|\\ &\le C\frac{|x-x'|}{\ell(Q_{x_0,k})}\frac{|y-y'|}{\ell(Q_{y_0,k})}\frac{1}{(\ell(Q_{x,k}) + \ell(Q_{y,k}) + |x-y|)^n}.\end{aligned}$$

Proof. Let $S_k(x,y)$ for $k \in \mathbb{Z}$ be the same as in (2.2). Tolsa in [27] has proved that $S_k(x,y)$ satisfy (a), (b), (c), (d) and (e); see [27] for the details. We only need to verify that Tolsa's approximations to the identity satisfy the second difference smoothness condition in (2.3).

Let $Q_{x,k}$, $Q_{x',k}$ and $Q_{y,k}$ be the transit cubes as in Definition 2.2. To verify (2.3), by (2.2), we have

$$\begin{aligned}&[S_k(x,y) - S_k(x',y)] - [S_k(x,y') - S_k(x',y')]\\ &= \int_{\mathbb{R}^d}\left[M_k(x)\widetilde{S}_k(x,z) - M_k(x')\widetilde{S}_k(x',z)\right]W_k(z)\\ &\times\left[\widetilde{S}_k(y,z)M_k(y) - \widetilde{S}_k(y',z)M_k(y')\right]\,d\mu(z)\\ &= [M_k(x) - M_k(x')]\,M_k(y)\left\{\int_{\mathbb{R}^d}\widetilde{S}_k(x,z)W_k(z)\left[\widetilde{S}_k(y,z) - \widetilde{S}_k(y',z)\right]\,d\mu(z)\right\}\\ &+ [M_k(x) - M_k(x')]\,[M_k(y) - M_k(y')]\left\{\int_{\mathbb{R}^d}\widetilde{S}_k(x,z)W_k(z)\widetilde{S}_k(y',z)\,d\mu(z)\right\}\\ &+ M_k(x')\,[M_k(y) - M_k(y')]\left\{\int_{\mathbb{R}^d}\left[\widetilde{S}_k(x,z) - \widetilde{S}_k(x',z)\right]W_k(z)\widetilde{S}_k(y',z)\,d\mu(z)\right\}\\ &+ M_k(x')M_k(y)\left\{\int_{\mathbb{R}^d}\left[\widetilde{S}_k(x,z) - \widetilde{S}_k(x',z)\right]W_k(z)\left[\widetilde{S}_k(y,z) - \widetilde{S}_k(y',z)\right]\,d\mu(z)\right\}\\ &= B_1 + B_2 + B_3 + B_4.\end{aligned}$$

It was proved by Tolsa in [27, pp. 78-79] that for x, $x' \in Q_{x_0,k}$,

$$(2.4)\qquad |M_k(x) - M_k(x')| \le C\frac{|x-x'|}{\ell(Q_{x_0,k})}$$

and

$$\begin{aligned}(2.5)\qquad &\int_{\mathbb{R}^d}\left|\widetilde{S}_k(y,z) - \widetilde{S}_k(y',z)\right|\widetilde{S}_k(x,z)\,d\mu(z)\\ &\le C\frac{|y-y'|}{\ell(Q_{y_0,k})}\frac{1}{(\ell(Q_{x,k}) + \ell(Q_{y,k}) + |x-y|)^n}.\end{aligned}$$

From the fact that for $z \in \mathrm{supp}\,(\mu)$ satisfying $Q_{z,k} \ne \mathbb{R}^d$,

$$(2.6)\qquad 0 \le W_k(z) \le 6 \quad\text{and}\quad 2/3 \le M_k(z) \le 4$$

(see Lemma 5.1 in [27]), the estimates (2.4) and (2.5), it follows that

$$(2.7)\qquad |B_1| \le C\frac{|x-x'|}{\ell(Q_{x_0,k})}\frac{|y-y'|}{\ell(Q_{y_0,k})}\frac{1}{(\ell(Q_{x,k}) + \ell(Q_{y,k}) + |x-y|)^n},$$

which is a desired estimate.

To estimate B_2, we first verify that

$$\int_{\mathbb{R}^d} \widetilde{S}_k(x,z)\widetilde{S}_k(y',z)\,d\mu(z) \le \frac{C}{(\ell(Q_{x,k})+\ell(Q_{y,k})+|x-y|)^n}. \tag{2.8}$$

By Lemma 4.4(b) and (c) in [27] and noting that $Q_{x,k}\subset \check{Q}^1_{x,k}$ (see Lemma 2.2), we have

$$0\le \widetilde{S}_k(x,z)=\varphi_{z,k}(x)\le \frac{C}{\ell(\check{Q}^1_{x,k})^n}\le \frac{C}{\ell(Q_{x,k})^n}. \tag{2.9}$$

On the other hand, from Lemma 2.3, it is easy to deduce that $y\in Q_{y_0,k}$ implies that $Q_{y,k}\subset \check{\check{Q}}^1_{y_0,k}$. Moreover, the fact that $Q_{y',k}\subset \check{\check{Q}}^1_{y',k}$ and $y'\in Q_{y_0,k}\subset \check{\check{Q}}^1_{y_0,k}$ imply that $\check{\check{Q}}^1_{y',k}\cap \check{\check{Q}}^1_{y_0,k}\neq\emptyset$, which together with Lemma 2.3 further indicates that $\check{\check{Q}}^1_{y_0,k}\subset \check{Q}^1_{y',k}$. Thus,

$$Q_{y_0,k}\cup Q_{y,k}\subset \check{Q}^1_{y',k}. \tag{2.10}$$

The fact (2.10) and the estimate (2.9) yield

$$0\le \widetilde{S}_k(y',z)=\varphi_{z,k}(y')\le \frac{C}{\ell(\check{Q}^1_{y',k})^n}\le \frac{C}{\ell(Q_{y,k})^n}. \tag{2.11}$$

The estimates (2.9) and (2.11), together with the fact that for all $x\in \operatorname{supp}(\mu)$,

$$\int_{\mathbb{R}^d} \widetilde{S}_k(x,z)\,d\mu(z)\le C \tag{2.12}$$

(see Lemma 4.5 in [27]), tell us that

$$\int_{\mathbb{R}^d} \widetilde{S}_k(x,z)\widetilde{S}_k(y',z)\,d\mu(z)\le C\min\left\{\frac{1}{\ell(Q_{x,k})^n},\frac{1}{\ell(Q_{y,k})^n}\right\}. \tag{2.13}$$

Lemma 4.4(b) and (c) in [27] also imply that

$$\widetilde{S}_k(x,z)\le \frac{C}{|x-z|^n} \quad\text{and}\quad \widetilde{S}_k(y',z)=\varphi_{z,k}(y')\le \frac{C}{\left(\ell(\check{Q}^1_{y',k})+|y'-z|\right)^n}, \tag{2.14}$$

which, together with (2.12), further yields

$$\begin{aligned}
&\int_{\mathbb{R}^d} \widetilde{S}_k(x,z)\widetilde{S}_k(y',z)\,d\mu(z) \\
&\le C\int_{|x-z|\ge|x-y|/2} \frac{1}{|x-z|^n}\widetilde{S}_k(y',z)\,d\mu(z) \\
&\quad +C\int_{|x-z|<|x-y|/2} \widetilde{S}_k(x,z)\frac{1}{\left(\ell(\check{Q}^1_{y',k})+|y'-z|\right)^n}\,d\mu(z) \\
&\le \frac{C}{|x-y|^n}\int_{|x-z|<|x-y|/2} \widetilde{S}_k(x,z)\frac{1}{(\ell(Q_{y,k})+|y-z|)^n}\,d\mu(z) \\
&\le \frac{C}{|x-y|^n},
\end{aligned} \tag{2.15}$$

where we used the fact that

$$\ell(Q_{y,k})+|y-z|\le C\left(\ell(\check{Q}^1_{y',k})+|y'-z|\right). \tag{2.16}$$

This can be deduced from (2.10) and

$$|y-z|\le|y-y'|+|y'-z|\le\sqrt{d}\,\ell(Q_{y_0,k})+|y'-z|\le\sqrt{d}\left(\ell(\check{Q}^1_{y',k})+|y'-z|\right),$$

since $y,\ y' \in Q_{y_0,k}$. The estimates (2.13) and (2.15) show (2.8). The estimates (2.4), (2.6) and (2.8) yield

$$|B_2| \le C \frac{|x-x'|}{\ell(Q_{x_0,k})} \frac{|y-y'|}{\ell(Q_{y_0,k})} \frac{1}{(\ell(Q_{x,k})+\ell(Q_{y,k})+|x-y|)^n}, \tag{2.17}$$

which is a desired estimate.

To estimate B_3, noting that for all $w \in Q_{x_0,k}$,

$$\left|\varphi'_{z,k}(w)\right| \le \frac{C}{\left(\ell(\check{Q}^1_{x_0,k})+|w-z|\right)^{n+1}} \tag{2.18}$$

(see (33) in [27]), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left|\widetilde{S}_k(x,z)-\widetilde{S}_k(x',z)\right| \widetilde{S}_k(y',z)\,d\mu(z)\\
&\le C\int_{\mathbb{R}^d} \frac{|x-x'|}{\left(\ell(\check{Q}^1_{x_0,k})+|x-z|\right)^{n+1}} \widetilde{S}_k(y',z)\,d\mu(z)\\
&\le C\int_{|z-y|\ge|x-y|/2} \frac{|x-x'|}{\left(\ell(\check{Q}^1_{x_0,k})+|x-z|\right)^{n+1}} \widetilde{S}_k(y',z)\,d\mu(z)\\
&\quad +C\int_{|z-y|<|x-y|/2} \frac{|x-x'|}{\left(\ell(\check{Q}^1_{x_0,k})+|x-z|\right)^{n+1}} \widetilde{S}_k(y',z)\,d\mu(z)\\
&= I_1+I_2.
\end{aligned}$$

For I_1, from (2.14) and (2.16), it follows that

$$\begin{aligned}
I_1 &\le C\int_{|z-y|\ge|x-y|/2} \frac{|x-x'|}{\left(\ell(\check{Q}^1_{x_0,k})+|x-z|\right)^{n+1}} \frac{1}{\left(\ell(\check{Q}^1_{y',k})+|y'-z|\right)^{n}}\,d\mu(z)\\
&\le C\int_{|z-y|\ge|x-y|/2} \frac{|x-x'|}{\left(\ell(\check{Q}^1_{x_0,k})+|x-z|\right)^{n+1}} \frac{1}{(\ell(Q_{y,k})+|y-z|)^{n}}\,d\mu(z)\\
&\le C\frac{1}{(\ell(Q_{y,k})+|x-y|)^n}\int_{\mathbb{R}^d} \frac{|x-x'|}{\left(\ell(\check{Q}^1_{x_0,k})+|x-z|\right)^{n+1}}\,d\mu(z)\\
&\le C\frac{|x-x'|}{\ell(Q_{x_0,k})}\frac{1}{(\ell(Q_{y,k})+|x-y|)^n}.
\end{aligned} \tag{2.19}$$

On the other hand, $x \in Q_{x_0,k}$ and Lemma 2.3 lead to $Q_{x,k} \subset \check{Q}^1_{x_0,k}$. This fact, $Q_{x_0,k} \subset \check{Q}^1_{x_0,k}$ and (2.12) imply that

$$\begin{aligned}
I_1 &\le C\frac{|x-x'|}{\ell(Q_{x_0,k})}\frac{1}{\ell(Q_{x,k})^n}\int_{\mathbb{R}^d} \widetilde{S}_k(y',z)\,d\mu(z)\\
&\le C\frac{|x-x'|}{\ell(Q_{x_0,k})}\frac{1}{\ell(Q_{x,k})^n}.
\end{aligned} \tag{2.20}$$

The estimates (2.19) and (2.20) yield

$$I_1 \le C\frac{|x-x'|}{\ell(Q_{x_0,k})}\frac{1}{(\ell(Q_{x,k})+\ell(Q_{y,k})+|x-y|)^n}. \tag{2.21}$$

For I_2, noting that it was proved by Tolsa in [27, p. 79] that

$$\ell(Q_{x,k})+\ell(Q_{y,k})\le C\left(\ell(\check{Q}^1_{x_0,k})+|x-y|\right), \tag{2.22}$$

by (2.12), we then have

$$\begin{aligned} I_2 &\le C\int_{|z-y|<|x-y|/2}\frac{|x-x'|}{\left(\ell(\check{Q}^1_{x_0,k})+|x-z|\right)^{n+1}}\widetilde{S}_k(y',z)\,d\mu(z)\\ &\le C\frac{|x-x'|}{\ell(Q_{x_0,k})}\frac{1}{\left(\ell(\check{Q}^1_{x_0,k})+|x-y|\right)^{n}}\\ &\le C\frac{|x-x'|}{\ell(Q_{x_0,k})}\frac{1}{(\ell(Q_{x,k})+\ell(Q_{y,k})+|x-y|)^{n}}. \end{aligned}\tag{2.23}$$

Combining (2.21), (2.23), (2.4) and (2.6) yields

$$|B_3|\le C\frac{|x-x'|}{\ell(Q_{x_0,k})}\frac{|y-y'|}{\ell(Q_{y_0,k})}\frac{1}{(\ell(Q_{x,k})+\ell(Q_{y,k})+|x-y|)^{n}}. \tag{2.24}$$

Finally, by (2.18) and (2.6), we have

$$\begin{aligned} |B_4| &\le C\int_{\mathbb{R}^d}\frac{|x-x'|}{\left(\ell(\check{Q}^1_{x_0,k})+|x-z|\right)^{n+1}}\frac{|y-y'|}{\left(\ell(\check{Q}^1_{y_0,k})+|y-z|\right)^{n+1}}\,d\mu(z)\\ &= C\int_{|y-z|\ge|x-y|/2}\frac{|x-x'|}{\left(\ell(\check{Q}^1_{x_0,k})+|x-z|\right)^{n+1}}\frac{|y-y'|}{\left(\ell(\check{Q}^1_{y_0,k})+|y-z|\right)^{n+1}}\,d\mu(z)\\ &\quad+C\int_{|y-z|<|x-y|/2}\frac{|x-x'|}{\left(\ell(\check{Q}^1_{x_0,k})+|x-z|\right)^{n+1}}\frac{|y-y'|}{\left(\ell(\check{Q}^1_{y_0,k})+|y-z|\right)^{n+1}}\,d\mu(z)\\ &= B_4^1+B_4^2. \end{aligned}$$

An easy computation implies that

$$\begin{aligned} B_4^1 &\le C\frac{|y-y'|}{\left(\ell(\check{Q}^1_{y_0,k})+|x-y|\right)^{n+1}}\int_{\mathbb{R}^d}\frac{|x-x'|}{\left(\ell(\check{Q}^1_{x_0,k})+|x-z|\right)^{n+1}}\,d\mu(z)\\ &\le C\frac{|x-x'|}{\ell(Q_{x_0,k})}\frac{|y-y'|}{\ell(Q_{y_0,k})}\frac{1}{\left(\ell(\check{Q}^1_{y_0,k})+|x-y|\right)^{n}}. \end{aligned}$$

Since $y\in Q_{y_0,k}$, then $Q_{y,k}\cap Q_{y_0,k}\neq\emptyset$ which, by Lemma 2.3, implies that

$$Q_{y,k}\subset\check{\check{Q}}^1_{y_0,k}\subset\check{Q}^1_{y_0,k}.$$

Thus,

$$\ell(Q_{y,k})\le\ell(\check{Q}^1_{y_0,k}). \tag{2.25}$$

Therefore,

$$\ell(Q_{y,k})+|x-y|\le C\left(\ell(\check{Q}^1_{y_0,k})+|x-y|\right).$$

If $|x-y|\le\ell(Q_{x,k})/2$, then $y\in Q_{x,k}$ and $Q_{x,k}\cap Q_{y,k}\neq\emptyset$, and hence, $Q_{x,k}\cap\check{\check{Q}}^1_{y_0,k}\neq\emptyset$. By Lemma 2.3 again, we obtain $Q_{x,k}\subset\check{Q}^1_{y_0,k}$. That is, $\ell(Q_{x,k})\le\ell(\check{Q}^1_{y_0,k})$. From this, it follows that

$$\ell(Q_{x,k})+\ell(Q_{y,k})+|x-y|\le C\left(\ell(\check{Q}^1_{y_0,k})+|x-y|\right).$$

By (2.25), we can easily see that this is also true if $|x-y| > \ell(Q_{x,k})/2$. Thus, we always have

$$B_4^1 \leq C\frac{|x-x'|}{\ell(Q_{x_0,k})}\frac{|y-y'|}{\ell(Q_{y_0,k})}\frac{1}{(\ell(Q_{x,k})+\ell(Q_{y,k})+|x-y|)^n}, \tag{2.26}$$

which is a desired estimate. We now estimate B_4^2. Again an easy computation, (2.25) and (2.22) indicate that

$$\begin{aligned} B_4^2 &\leq C\frac{|x-x'|}{\Big(\ell(\check{Q}^1_{x_0,k})+|x-y|\Big)^{n+1}}\int_{\mathbb{R}^d}\frac{|y-y'|}{\Big(\ell(\check{Q}^1_{y_0,k})+|y-z|\Big)^{n+1}}\,d\mu(z) \\ &\leq C\frac{|x-x'|}{\Big(\ell(\check{Q}^1_{x_0,k})+|x-y|\Big)^{n+1}}\frac{|y-y'|}{\ell(\check{Q}^1_{y_0,k})} \\ &\leq C\frac{|x-x'|}{\ell(Q_{x_0,k})}\frac{|y-y'|}{\ell(Q_{y_0,k})}\frac{1}{\Big(\ell(\check{Q}^1_{x_0,k})+|x-y|\Big)^{n}} \\ &\leq C\frac{|x-x'|}{\ell(Q_{x_0,k})}\frac{|y-y'|}{\ell(Q_{y_0,k})}\frac{1}{(\ell(Q_{x,k})+\ell(Q_{y,k})+|x-y|)^n}. \end{aligned} \tag{2.27}$$

Combining the estimates (2.26) and (2.27), we obtain

$$|B_4| \leq C\frac{|x-x'|}{\ell(Q_{x_0,k})}\frac{|y-y'|}{\ell(Q_{y_0,k})}\frac{1}{(\ell(Q_{x,k})+\ell(Q_{y,k})+|x-y|)^n}. \tag{2.28}$$

Now (2.3) follows from the estimates (2.7), (2.17), (2.24) and (2.28), and hence, we have completed the proof of Lemma 2.4. □

Remark 2.2. Taking the (formal) Definition 2.8 of the kernels $\widetilde{S}_k(x,y)$, it is easily seen that the properties of the kernels $S_k(x,y)$ in (a), (b), (c), (d), (e) and (f) of Lemma 2.4 also hold without assuming that $Q_{x,k}$, $Q_{x',k}$ and $Q_{y,k}$ are transit cubes; see also [27]. In what follows, without loss of generality, we will always assume that $Q_{x,k}$, $Q_{x',k}$ and $Q_{y,k}$ are transit cubes.

Definition 2.9. A sequence of operators $\{S_k\}_{k\in\mathbb{Z}}$ is said to be an approximation to the identity associated to non-doubling measure μ if $\{S_k(x,y)\}_{k=-\infty}^{\infty}$, the kernels of $\{S_k\}_{k=-\infty}^{\infty}$, satisfy conditions (a), (b), (c), (d), (e) and (f) of Lemma 2.4.

Now, let $\{S_k\}_{k\in\mathbb{Z}}$ be an approximation to the identity as in Definition 2.9 and set $D_k = S_k - S_{k-1}$ for $k\in\mathbb{Z}$. Following [3] and [27], based on Coifman's idea, we can write

$$I = T_N + R_N, \tag{2.29}$$

where $T_N = \sum_{|k-j|\leq N} D_kD_j$ and $R_N = \sum_{|k-j|>N} D_kD_j$; see also (1.3).

If we set $D_k^N = \sum_{|j|\leq N} D_{k+j}$ for $k\in\mathbb{Z}$, then we can also write

$$T_N = \sum_{k\in\mathbb{Z}} D_k^N D_k.$$

As mentioned in the Introduction, the following result is a crucial observation of this paper.

Theorem 2.1. *Let $\{S_k\}_{k\in\mathbb{Z}}$, $\{A_k\}_{k\in\mathbb{Z}}$ and $\{P_k\}_{k\in\mathbb{Z}}$ be approximations to the identity as in Definition* 2.9. *Set $D_k = S_k - S_{k-1}$, $G_k = A_k - A_{k-1}$ and $E_k = P_k - P_{k-1}$ for $k \in \mathbb{Z}$. Then, if $1 \le p,\ q \le \infty$ and $|s| < \theta$, where θ is the same as in Definition* 2.5, *for all $f \in L^2(\mu)$,*

$$\left\{\sum_{j=-\infty}^{\infty} 2^{jsq}\|E_jR_Nf\|^q_{L^p(\mu)}\right\}^{1/q} \le C_1\left(2^{-N(s+2\nu\theta)} + 2^{-N(2\nu\theta-s)}\right)\left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\|G_kf\|^q_{L^p(\mu)}\right\}^{1/q} \tag{2.30}$$

with C_1 independent of N and f, and $\nu \in (0, 1/2)$ with $|s| < 2\nu\theta$. Moreover, if we choose $N \in \mathbb{N}$ such that

$$C_1\left(2^{-N(s+2\nu\theta)} + 2^{-N(2\nu\theta-s)}\right) < 1, \tag{2.31}$$

then for all $f \in L^2(\mu)$,

$$\left\{\sum_{j=-\infty}^{\infty} 2^{jsq}\|E_jT_N^{-1}f\|^q_{L^p(\mu)}\right\}^{1/q} \le C\left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\|G_kf\|^q_{L^p(\mu)}\right\}^{1/q}, \tag{2.32}$$

where C is independent of f, and T_N and R_N are the same as in (2.29).

To show Theorem 2.1, we need the following lemma.

Lemma 2.5. *Let θ be the same as in Definition* 2.5 *and let $\{S_k\}_{k\in\mathbb{Z}}$ and $\{P_k\}_{k\in\mathbb{Z}}$ be two approximations to the identity as in Definition* 2.9. *Set $D_k = S_k - S_{k-1}$ and $E_k = P_k - P_{k-1}$ for $k \in \mathbb{Z}$. Then,*

(i) $\operatorname{supp}(E_jD_k)(x,\cdot) \subset Q_{x,\min(j,k)-3}$ *and* $\operatorname{supp}(E_jD_k)(\cdot,y) \subset Q_{y,\min(j,k)-3}$ *for all $j,\ k \in \mathbb{Z}$ and all $x,\ y \in \operatorname{supp}(\mu)$;*

(ii) *for all $x,\ y \in \operatorname{supp}(\mu)$ and all $j,\ k \in \mathbb{Z}$,*

$$|(E_jD_k)(x,y)| \le C2^{-2|j-k|\theta}\frac{1}{(\ell(Q_{x,\min(j,k)+1}) + \ell(Q_{y,\min(j,k)+1}) + |x-y|)^n};$$

(iii) *for $p \in [1,\infty]$ and $j,\ k \in \mathbb{Z}$,*

$$\|E_jD_k\|_{L^p(\mu)\to L^p(\mu)} \le C_2 2^{-2|j-k|\theta},$$

where $C_2 > 0$ is a constant depends on p, but not on j and k.

Proof. The proof of this lemma is essentially contained in the proof of Lemma 6.1 in [27]. For the reader's convenience, we give some details.

We first remark that one can deduce (iii) from (ii) and (i); see [27]. Thus, it suffices to verify (i) and (ii). Without loss of generality, we may assume that all the cubes $Q_{x,k}$, $x \in \operatorname{supp}(\mu)$ and $k \in \mathbb{Z}$, are transit cubes; see [27, pp. 80-81]. We now prove (i) and (ii) by considering several cases.

Case 1. $j \ge k + 2$. In this case, completely similar to the proof of (34) in [27], we have

$$|(E_jD_k)(x,y)| \le C2^{-2|j-k|\theta}\frac{1}{(\ell(Q_{x,k}) + \ell(Q_{y,k}) + |x-y|)^n}.$$

Lemma 2.4 and Lemma 2.3 also tell us that $\operatorname{supp}(E_jD_k)(x,\cdot) \subset Q_{x,k-3}$ and

$$\operatorname{supp}(E_jD_k)(\cdot,y) \subset Q_{y,k-3}.$$

Case 2. $k-1 \le j \le k$. In this case, it suffices to prove that

$$|(E_jD_k)(x,y)| \le \frac{C}{(\ell(Q_{x,j})+\ell(Q_{y,j+1})+|x-y|)^n}. \tag{2.33}$$

To verify (2.33), it suffices to verify that

$$|(E_jD_k)(x,y)| \le \frac{C}{\ell(Q_{x,j})^n}, \tag{2.34}$$

$$|(E_jD_k)(x,y)| \le \frac{C}{\ell(Q_{y,j+1})^n} \tag{2.35}$$

and

$$|(E_jD_k)(x,y)| \le \frac{C}{|x-y|^n}. \tag{2.36}$$

The size condition of E_j (Lemma 2.4(d)) and the integral condition of D_k (Lemma 2.4(b)) immediately yield that

$$\begin{aligned}
|(E_jD_k)(x,y)| &= \left|\int_{\mathbb{R}^d} E_j(x,z)D_k(z,y)\,d\mu(y)\right| \\
&\le C\int_{\mathbb{R}^d} \frac{1}{(\ell(Q_{x,j})+\ell(Q_{y,j})+|x-z|)^n}|D_k(z,y)|\,d\mu(z) \\
&\le \frac{C}{\ell(Q_{x,j})^n},
\end{aligned}$$

which is (2.34).

Since $k \le j+1$, by Lemma 2.2, $Q_{y,j+1} \subset Q_{y,k}$ and, thus, $\ell(Q_{y,k}) \ge \ell(Q_{y,j+1})$. This fact together with Lemma 2.4 tell us that

$$\begin{aligned}
|(E_jD_k)(x,y)| &= \left|\int_{\mathbb{R}^d} E_j(x,z)D_k(z,y)\,d\mu(y)\right| \\
&\le C\int_{\mathbb{R}^d} |E_j(x,z)|\frac{1}{(\ell(Q_{z,k})+\ell(Q_{y,k})+|z-y|)^n}\,d\mu(z) \\
&\le \frac{C}{\ell(Q_{y,j+1})^n},
\end{aligned}$$

which is (2.35).

On the other hand, by Lemma 2.4, we can also obtain

$$\begin{aligned}
&|(E_jD_k)(x,y)| \\
&= \left|\int_{\mathbb{R}^d} E_j(x,z)D_k(z,y)\,d\mu(y)\right| \\
&\le C\int_{|x-z|\ge\frac{|x-y|}{2}} \frac{1}{(\ell(Q_{x,j})+\ell(Q_{y,j})+|x-z|)^n}|D_k(z,y)|\,d\mu(z) \\
&\quad +C\int_{|x-z|<\frac{|x-y|}{2}} |E_j(x,z)|\frac{1}{(\ell(Q_{z,k})+\ell(Q_{y,k})+|z-y|)^n}\,d\mu(z) \\
&\le \frac{C}{|x-y|^n}\left[\int_{\mathbb{R}^d}|D_k(z,y)|\,d\mu(z)+\int_{\mathbb{R}^d}|E_j(x,z)|\,d\mu(z)\right] \\
&\le \frac{C}{|x-y|^n},
\end{aligned}$$

which is (2.36). Thus, (2.33) is true.

It is also easy to see that $\operatorname{supp}(E_jD_k)(x,\cdot)\subset Q_{x,j-3}$ and $\operatorname{supp}(E_jD_k)(\cdot,y)\subset Q_{y,j-3}$ by Lemma 2.3 and Lemma 2.4.

Case 3. $j=k+1$. In this case, by Lemma 2.3 and Lemma 2.4, we have

$$\operatorname{supp}(E_jD_k)(x,\cdot)\subset Q_{x,k-3}$$

and $\operatorname{supp}(E_jD_k)(\cdot,y)\subset Q_{y,k-3}$. An argument similar to that of (2.35) tells us that

$$|(E_jD_k)(x,y)|\le \frac{C}{(\ell(Q_{x,k+1})+\ell(Q_{y,k})+|x-y|)^n}.$$

Case 4. $j\le k-2$. In this case, by Lemma 2.3 and Lemma 2.4, we have

$$\operatorname{supp}(E_jD_k)(x,\cdot)\subset Q_{x,j-3}$$

and $\operatorname{supp}(E_jD_k)(\cdot,y)\subset Q_{y,j-3}$. By an argument similar to that of (34) in [27], we can show that

$$|(E_jD_k)(x,y)|\le C2^{-2|j-k|\theta}\frac{1}{(\ell(Q_{x,j})+\ell(Q_{y,j})+|x-y|)^n}.$$

This finishes the proof of Lemma 2.5.

Before we return to the proof of Theorem 2.1, by a result of Tolsa in [27], if N is big enough, then for all $f\in L^2(\mu)$, we have that

$$f=\sum_{k\in\mathbb{Z}}T_N^{-1}D_k^ND_k(f)=\sum_{k\in\mathbb{Z}}D_k^ND_kT_N^{-1}(f) \tag{2.37}$$

holds in the norm of $L^2(\mu)$. In fact, T_N^{-1} is bounded on $L^p(\mu)$ with $1<p<\infty$. The formula (2.37) is called the Calderón-type reproducing formula; see [27] for more details.

We now write T_N^{-1} as

$$T_N^{-1}=\sum_{l=0}^{\infty}(R_N)^l \tag{2.38}$$

in the operator norm of $L^2(\mu)$, and for $l\in\mathbb{N}$,

$$(R_N)^l=\sum_{|k_1-j_1|>N}D_{k_1}D_{j_1}\sum_{|k_2-j_2|>N}D_{k_2}D_{j_2}\cdots\sum_{|k_l-j_l|>N}D_{k_l}D_{j_l} \tag{2.39}$$

also in the operator norm of $L^2(\mu)$.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. We only need to show (2.30). The inequality (2.32) can be deduced from (2.30) by using (2.38) and (2.39).

To verify (2.30), the formulae (2.37), (2.38) and (2.39) with D_k replaced by G_k tell us that for all $j\in\mathbb{Z}$,

$$\begin{aligned} E_jR_Nf(x)=&\sum_{l=0}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\sum_{|m|>N}\sum_{k_1=-\infty}^{\infty}\sum_{|m_1|>N_1}\cdots\sum_{k_l=-\infty}^{\infty}\\ &\times\sum_{|m_l|>N_1}E_jD_iD_{i+m}G_{k_1}G_{k_1+m_1}\cdots G_{k_l}G_{k_l+m_l}G_k^{N_1}G_kf(x). \end{aligned} \tag{2.40}$$

By a technique used in the proof of the Cotlar-Stein lemma (see [22]) and Lemma 2.5, we obtain that for $p \in [1,\infty]$, there is a constant $C_2 > 0$ such that for all $f \in L^2(\mu)$,

$$\begin{aligned}
&\left\|E_jD_iD_{i+m}G_{k_1}G_{k_1+m_1}\cdots G_{k_l}G_{k_l+m_l}G_k^{N_1}G_kf\right\|_{L^p(\mu)} \\
&\le \|E_jD_i\|_{L^p(\mu)\to L^p(\mu)}\|D_{i+m}G_{k_1}\|_{L^p(\mu)\to L^p(\mu)}\cdots \\
&\quad\times\left\|G_{k_l+m_l}G_k^{N_1}\right\|_{L^p(\mu)\to L^p(\mu)}\|G_kf\|_{L^p(\mu)\to L^p(\mu)} \\
&\le CN_1C_2^l2^{-2\theta[|j-i|+|i+m-k_1|+\cdots+|k_l+m_l-k|]}\|G_kf\|_{L^p(\mu)}
\end{aligned} \tag{2.41}$$

and

$$\begin{aligned}
&\left\|E_jD_iD_{i+m}G_{k_1}G_{k_1+m_1}\cdots G_{k_l}G_{k_l+m_l}G_k^{N_1}G_kf\right\|_{L^p(\mu)} \\
&\le C\|E_j\|_{L^p(\mu)\to L^p(\mu)}\|D_iD_{i+m}\|_{L^p(\mu)\to L^p(\mu)}\cdots \\
&\quad\times\|G_{k_l}G_{k_l+m_l}\|_{L^p(\mu)\to L^p(\mu)}\left\|G_k^{N_1}\right\|_{L^p(\mu)\to L^p(\mu)}\|G_kf\|_{L^p(\mu)} \\
&\le CN_1C_2^l2^{-2\theta[|m|+|m_1|+\cdots+|m_l|]}\|G_kf\|_{L^p(\mu)}.
\end{aligned} \tag{2.42}$$

Here we also used the fact that $\|E_j\|_{L^p(\mu)\to L^p(\mu)} \le C$ uniformly on j and

$$\left\|G_k^{N_1}\right\|_{L^p(\mu)\to L^p(\mu)} \le CN_1$$

uniformly on k with C independent of N_1. The geometric means of (2.41) and (2.42) yields that

$$\begin{aligned}
&\left\|E_jD_iD_{i+m}G_{k_1}G_{k_1+m_1}\cdots G_{k_l}G_{k_l+m_l}G_k^{N_1}G_kf\right\|_{L^p(\mu)} \\
&\le CN_1C_2^l2^{-2\theta(1-\nu)[|j-i|+|i+m-k_1|+\cdots+|k_l+m_l-k|]} \\
&\quad\times 2^{-2\theta\nu[|m|+|m_1|+\cdots+|m_l|]}\|G_kf\|_{L^p(\mu)}.
\end{aligned} \tag{2.43}$$

From (2.40) and (2.43), it follows that

$$\begin{aligned}
&\left\{\sum_{j=-\infty}^{\infty}2^{jsq}\|E_jR_Nf\|_{L^p(\mu)}^q\right\}^{1/q} \\
&\le\left\{\sum_{j=-\infty}^{\infty}2^{jsq}\left[\sum_{l=0}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\sum_{|m|>N}\sum_{k_1=-\infty}^{\infty}\sum_{|m_1|>N_1}\cdots\sum_{k_l=-\infty}^{\infty}\right.\right. \\
&\quad\left.\left.\times\sum_{|m_l|>N_1}\|E_jD_iD_{i+m}G_{k_1}G_{k_1+m_1}\cdots G_{k_l}G_{k_l+m_l}G_k^{N_1}G_kf\|_{L^p(\mu)}\right]^q\right\}^{1/q} \\
&\le C\sum_{l=0}^{\infty}C_2^l\left\{\sum_{j=-\infty}^{\infty}\left[\sum_{i=-\infty}^{\infty}2^{(j-i)s-2\theta(1-\nu)|j-i|}\sum_{|m|>N}\sum_{k_1=-\infty}^{\infty}\sum_{|m_1|>N_1}\cdots\sum_{k_l=-\infty}^{\infty}\right.\right. \\
&\quad\times\sum_{|m_l|>N_1}\sum_{k=-\infty}^{\infty}2^{is}2^{-2\theta(1-\nu)[|i+m-k_1|+\cdots+|k_l+m_l-k|]} \\
&\quad\left.\left.\times 2^{-2\theta\nu[|m|+|m_1|+\cdots+|m_l|]}\|G_kf\|_{L^p(\mu)}\right]^q\right\}^{1/q}.
\end{aligned}$$

Using the fact that $|s| < 2(1-\nu)\theta$ and the Hölder inequality on the summation of i, and then exchanging the order of the summation on i with the summation on j, we then know that the last quantity in the above inequality can be controlled by

$$\leq C \sum_{l=0}^{\infty} C_2^l \left\{ \sum_{j=-\infty}^{\infty} \left(\left[\sum_{i=-\infty}^{\infty} 2^{(j-i)s-2\theta(1-\nu)|j-i|} \left(\sum_{|m|>N} \sum_{k_1=-\infty}^{\infty} \sum_{|m_1|>N_1} \cdots \sum_{k_l=-\infty}^{\infty} \right.\right.\right.\right.$$
$$\times \sum_{|m_l|>N_1} \sum_{k=-\infty}^{\infty} 2^{is} 2^{-2\theta(1-\nu)[|i+m-k_1|+\cdots+|k_l+m_l-k|]}$$
$$\left.\left. \times 2^{-2\theta\nu[|m|+|m_1|+\cdots+|m_l|]} \|G_k f\|_{L^p(\mu)} \right)^q \right]^{1/q}$$
$$\left.\left. \times \left[\sum_{i=-\infty}^{\infty} 2^{(j-i)s-2\theta(1-\nu)|j-i|} \right]^{1/q'} \right)^q \right\}^{1/q}$$
$$\leq C \sum_{l=0}^{\infty} C_2^l \left\{ \sum_{i=-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} 2^{(j-i)s-2\theta(1-\nu)|j-i|} \right) \left[\sum_{|m|>N} \sum_{k_1=-\infty}^{\infty} \sum_{|m_1|>N_1} \cdots \sum_{k_l=-\infty}^{\infty} \right.\right.$$
$$\times \sum_{|m_{l-1}|>N_1} \sum_{k=-\infty}^{\infty} \sum_{|m_l|>N_1} 2^{is} 2^{-2\theta(1-\nu)[|i+m-k_1|+\cdots+|k_l+m_l-k|]}$$
$$\left.\left. \times 2^{-2\theta\nu[|m|+|m_1|+\cdots+|m_l|]} \|G_k f\|_{L^p(\mu)} \right]^q \right\}^{1/q}$$
$$\leq C \sum_{l=0}^{\infty} C_2^l \left\{ \sum_{i=-\infty}^{\infty} \left[\sum_{k_1=-\infty}^{\infty} 2^{(i+m-k_1)s-2\theta(1-\nu)|i+m-k_1|} \sum_{|m|>N} \sum_{k_2=-\infty}^{\infty} \sum_{|m_1|>N_1} \cdots \right.\right.$$
$$\times \sum_{k_l=-\infty}^{\infty} \sum_{|m_{l-1}|>N_1} \sum_{k=-\infty}^{\infty} \sum_{|m_l|>N_1} 2^{(k_1-m)s} 2^{-2\theta(1-\nu)[|k_1+m_1-k_2|+\cdots+|k_l+m_l-k|]}$$
$$\left.\left. \times 2^{-2\theta\nu[|m|+|m_1|+\cdots+|m_l|]} \|G_k f\|_{L^p(\mu)} \right]^q \right\}^{1/q}.$$

For the summation on k_1 and the summation on i, repeating the same procedure as that on the summation on i and the summation on j, that is, using the fact that $|s| < 2(1-\nu)\theta$ and the Hölder inequality on the summation of k_1 and then exchanging the order of the summation on k_1 with the summation on i, we find that the last term in the above inequality is dominated by

$$\leq C \sum_{l=0}^{\infty} C_2^l \left\{ \sum_{k_1=-\infty}^{\infty} \left[\sum_{|m|>N} 2^{-ms-2\theta\nu|m|} \sum_{k_2=-\infty}^{\infty} \sum_{|m_1|>N_1} \cdots \right.\right.$$
$$\times \sum_{k_l=-\infty}^{\infty} \sum_{|m_{l-1}|>N_1} \sum_{k=-\infty}^{\infty} \sum_{|m_l|>N_1} 2^{k_1 s} 2^{-2\theta(1-\nu)[|k_1+m_1-k_2|+\cdots+|k_l+m_l-k|]}$$
$$\left.\left. \times 2^{-2\theta\nu[|m_1|+\cdots+|m_l|]} \|G_k f\|_{L^p(\mu)} \right]^q \right\}^{1/q}.$$

The fact $|s| < 2\theta\nu$ and the summation on m tell us that the above quantity can be controlled by

$$\leq \widetilde{C}_2 \left(2^{-N(s+2\theta\nu)} + 2^{-N(2\theta\nu-s)}\right) \sum_{l=0}^{\infty} C_2^l \left\{ \sum_{k_1=-\infty}^{\infty} \left[\sum_{k_2=-\infty}^{\infty} \sum_{|m_1|>N_1} \cdots \right.\right.$$
$$\times \sum_{k_l=-\infty}^{\infty} \sum_{|m_{l-1}|>N_1} \sum_{k=-\infty}^{\infty} \sum_{|m_l|>N_1} 2^{k_1 s} 2^{-2\theta(1-\nu)[|k_1+m_1-k_2|+\cdots+|k_l+m_l-k|]}$$
$$\left.\left. \times\, 2^{-2\theta\nu[|m_1|+\cdots+|m_l|]} \|G_k f\|_{L^p(\mu)} \right]^q \right\}^{1/q},$$

where $\widetilde{C}_2$ is a positive constant independent of N. Repeating this procedure l times, we finally obtain that the above term is dominated by

(2.44)

$$\leq \widetilde{C}_2 \left(2^{-N(s+2\theta\nu)} + 2^{-N(2\theta\nu-s)}\right)$$
$$\times \left\{ \sum_{l=0}^{\infty} \left(\widetilde{C}_2 C_2\right)^l \left(2^{-N_1(s+2\theta\nu)} + 2^{-N_1(2\theta\nu-s)}\right)^l \right\} \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} \|G_k f\|_{L^p(\mu)}^q \right\}^{1/q}.$$

If we now choose $N_1 \in \mathbb{N}$ large enough so that

$$\widetilde{C}_2 \left(2^{-N_1(s+2\theta\nu)} + 2^{-N_1(2\theta\nu-s)}\right) < 1$$

and then let

$$C_1 = \widetilde{C}_2 \left\{ \sum_{l=0}^{\infty} \left(\widetilde{C}_2 C_2\right)^l \left(2^{-N_1(s+2\theta\nu)} + 2^{-N_1(2\theta\nu-s)}\right)^l \right\},$$

we obtain (2.30).

This finishes the proof of Theorem 2.1.

We now use the approximation to the identity in Definition 2.9 to introduce the space of test functions.

Definition 2.10. Let $\{S_k\}_{k\in\mathbb{Z}}$ be an approximation to the identity as in Definition 2.9, $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}$ and let θ be the same as in Definition 2.5. For $|s| < \theta$, $1 \leq p,\ q \leq \infty$ and $f \in L^2(\mu)$, we define

$$\|f\|_{\dot{\mathcal{B}}^s_{pq}(\mu)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} \|D_k f\|_{L^p(\mu)}^q \right\}^{1/q}$$

and

$$\dot{\mathcal{B}}^s_{pq}(\mu) = \{f \in L^2(\mu) : \|f\|_{\dot{\mathcal{B}}^s_{pq}(\mu)} < \infty\}.$$

Applying Theorem 2.1, we can show that the space of test functions, $\dot{\mathcal{B}}^s_{pq}(\mu)$, in Definition 2.10 is independent of the chosen approximations to the identity.

Proposition 2.1. *Let θ be the same as in Definition 2.5. Let $\{S_k\}_{k\in\mathbb{Z}}$ and $\{P_k\}_{k\in\mathbb{Z}}$ be two approximations to the identity as in Definition 2.9. Set $D_k = S_k - S_{k-1}$ and*

$E_k = P_k - P_{k-1}$ for $k \in \mathbb{Z}$. Then for $|s| < \theta$ and $1 \le p,\ q \le \infty$, and all $f \in L^2(\mu)$,

$$\left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\|D_k f\|_{L^p(\mu)}^q\right\}^{1/q} \sim \left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\|E_k f\|_{L^p(\mu)}^q\right\}^{1/q}.$$

Proof. For given $|s| < \theta$, we choose $\nu \in (0, 1/2)$ such that $|s| < 2\nu\theta$. By (2.37), for any $j \in \mathbb{Z}$, we can write

$$E_j f(x) = \sum_{k=-\infty}^{\infty} E_j D_k^N D_k T_N^{-1}(f)(x),$$

where $N \in \mathbb{N}$ is large enough such that (2.31) holds.

Then the Minkowski inequality, Lemma 2.5 and Theorem 2.1 yield

$$\begin{aligned}
&\left\{\sum_{j=-\infty}^{\infty} 2^{jsq}\|E_j f\|_{L^p(\mu)}^q\right\}^{1/q}\\
&\le \left\{\sum_{j=-\infty}^{\infty} 2^{jsq}\left[\sum_{k=-\infty}^{\infty} \left\|E_j D_k^N D_k T_N^{-1}(f)\right\|_{L^p(\mu)}\right]^q\right\}^{1/q}\\
&\le C\left\{\sum_{j=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} 2^{(j-k)s-2\theta|j-k|}2^{ks}\left\|D_k T_N^{-1}(f)\right\|_{L^p(\mu)}\right]^q\right\}^{1/q}\\
&\le C\left\{\sum_{j=-\infty}^{\infty} \left[\left(\sum_{k=-\infty}^{\infty} 2^{(j-k)s-2\theta|j-k|}2^{ksq}\left\|D_k T_N^{-1}(f)\right\|_{L^p(\mu)}^q\right)^{1/q}\right.\right.\\
&\qquad\left.\left.\times\left(\sum_{k=-\infty}^{\infty} 2^{(j-k)s-2\theta|j-k|}\right)^{1/q'}\right]^q\right\}^{1/q}\\
&\le C\left\{\sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} 2^{(j-k)s-2\theta|j-k|}\right)2^{ksq}\left\|D_k T_N^{-1}(f)\right\|_{L^p(\mu)}^q\right\}^{1/q}\\
&= C\left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\left\|D_k T_N^{-1}(f)\right\|_{L^p(\mu)}^q\right\}^{1/q}\\
&\le C\left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\|D_k f\|_{L^p(\mu)}^q\right\}^{1/q}.
\end{aligned}$$

By symmetry, we have then finished the proof of Proposition 2.1.

The following theorem is one of the main results of this paper.

Theorem 2.2. *Let θ be the same as in Definition 2.5, and let $\{D_k\}_{k\in\mathbb{Z}}$ be the same as in Theorem 2.1. If $1 \le p \le \infty$ and $1 \le q < \infty$, then for all $f \in \dot{\mathcal{B}}_{pq}^s(\mu)$,*

$$f = \sum_{k\in\mathbb{Z}} D_k^N D_k T_N^{-1}(f) = \sum_{k\in\mathbb{Z}} T_N^{-1} D_k^N D_k(f) \tag{2.45}$$

holds in both the norm $\|\cdot\|_{\dot{B}^s_{pq}(\mu)}$ *and the norm* $\|\cdot\|_{\dot{B}^s_{p\infty}(\mu)}$. *Moreover, for all* $g \in \dot{\mathcal{B}}^s_{pq}(\mu)$ *with* $1 \le p,\ q < \infty$,

$$\langle f, g\rangle = \sum_{k\in\mathbb{Z}} \left\langle D_k D_k^N T_N^{-1}(f), g\right\rangle = \sum_{k\in\mathbb{Z}} \left\langle T_N^{-1} D_k D_k^N(f), g\right\rangle \tag{2.46}$$

holds for all $f \in \left(\dot{\mathcal{B}}^s_{pq}(\mu)\right)'$ *with* $1 \le p,\ q \le \infty$.

Proof. We only show the first equality in (2.45). The proof for the second equality in (2.45) is similar. The proof that (2.45) holds in the norm $\|\cdot\|_{\dot{B}^s_{p\infty}(\mu)}$ is easy by noting that $\dot{\mathcal{B}}^s_{pq}(\mu) \subset \dot{\mathcal{B}}^s_{p\infty}(\mu)$ for $1 \le q < \infty$, which is a simple result of the monotonicity of l^q; see the proof of Proposition 2.3.2/2 in [31, p. 47].

Let $f \in \dot{\mathcal{B}}^s_{pq}(\mu)$, $1 \le p \le \infty$ and $1 \le q < \infty$. Then, it suffices to show that

$$\lim_{L\to\infty} \left\| \sum_{|k|>L} D_k^N D_k T_N^{-1}(f) \right\|_{\dot{B}^s_{pq}(\mu)} = 0. \tag{2.47}$$

Lemma 2.5, the Minkowski inequality, the Hölder inequality and Theorem 2.1 lead to

$$\begin{aligned}
&\| \sum_{|k|>L} D_k^N D_k T_N^{-1}(f)\|_{\dot{B}^s_{pq}(\mu)} \\
&= \left\{ \sum_{j=-\infty}^{\infty} 2^{jsq} \|D_j \left(\sum_{|k|>L} D_k^N D_k T_N^{-1}(f)\right) \|^q_{L^p(\mu)} \right\}^{1/q} \\
&\le C \left\{ \sum_{j=-\infty}^{\infty} \left[\sum_{|k|>L} 2^{(j-k)s-2|j-k|\theta} 2^{ks} \|D_k T_N^{-1}(f)\|_{L^p(\mu)} \right]^q \right\}^{1/q} \\
&\le C \left\{ \sum_{j=-\infty}^{\infty} \left[\left(\sum_{|k|>L} 2^{(j-k)s-2|j-k|\theta} 2^{ksq} \|D_k T_N^{-1}(f)\|^q_{L^p(\mu)} \right)^{1/q} \right.\right. \\
&\qquad \left.\left. \times \left(\sum_{|k|>L} 2^{(j-k)s-2|j-k|\theta} \right)^{1/q'} \right]^q \right\}^{1/q} \\
&\le C \left\{ \sum_{|k|>L} \left[\sum_{j=-\infty}^{\infty} 2^{(j-k)s-2|j-k|\theta} \right] 2^{ksq} \|D_k T_N^{-1}(f)\|^q_{L^p(\mu)} \right\}^{1/q} \\
&\le C \left\{ \sum_{|k|>L} 2^{ksq} \|D_k T_N^{-1}(f)\|^q_{L^p(\mu)} \right\}^{1/q} \\
&\to 0
\end{aligned}$$

as $L \to \infty$, since $T_N^{-1}(f) \in \dot{\mathcal{B}}^s_{pq}(\mu)$. Thus, (2.47) holds and therefore, the first equality in (2.45) holds.

From (2.45) we can deduce the second equality in (2.46). In fact, for all $g \in \dot{\mathcal{B}}^s_{pq}(\mu)$ with $1 \le p,\ q < \infty$, then

$$\langle f, g\rangle = \left\langle f, \sum_{k\in\mathbb{Z}} D_k^N D_k T_N^{-1}(g)\right\rangle = \sum_{k\in\mathbb{Z}} \left\langle f, D_k^N D_k T_N^{-1}(g)\right\rangle,$$

where $f \in \left(\dot{\mathcal{B}}^s_{pq}(\mu)\right)'$ with $1 \le p, q \le \infty$.

To finish the proof, we only need to verify that for any $k \in \mathbb{Z}$,

$$\left\langle f, D_k^N D_k T_N^{-1}(g)\right\rangle = \left\langle D_k D_k^N T_N^{-1}(f), g\right\rangle. \tag{2.48}$$

To this end, for any $M > 0$, let $Q_{0,M}$ be the cube centered at the origin with the side length $2M$. Define

$$g_{k,M}(x) = \int_{Q_{0,M}} D_k^N(x,y)\left(D_k T_N^{-1}\right)(g)(y)\, d\mu(y).$$

We claim that

$$\lim_{M\to\infty} \left\|D_k^N D_k T_N^{-1}(g) - g_{k,M}\right\|_{\dot{B}^s_{pq}(\mu)} = 0. \tag{2.49}$$

In fact, Theorem 2.1 tells us that $T_N^{-1} g \in \dot{B}^s_{pq}(\mu)$ and Lemma 2.5 yields

$$\begin{aligned}
&\left\|D_k^N D_k T_N^{-1}(g) - g_{k,M}\right\|_{\dot{B}^s_{pq}(\mu)}\\
&= \left\{\sum_{l=-\infty}^{\infty} 2^{lsq}\left\|D_l\left[\int_{\mathbb{R}^d\setminus Q_{0,M}} D_k^N(\cdot,y)\left(D_k T_N^{-1}\right)(g)(y)\, d\mu(y)\right]\right\|^q_{L^p(\mu)}\right\}^{1/q}\\
&\le \left\{\sum_{l=-\infty}^{\infty} 2^{lsq}\left\|D_l D_k^N\right\|^q_{L^p(\mu)\to L^p(\mu)}\left[\int_{\mathbb{R}^d\setminus Q_{0,M}} \left|\left(D_k T_N^{-1}\right)(g)(y)\right|^p\, d\mu(y)\right]^{q/p}\right\}^{1/q}\\
&\le CN\left\{\sum_{l=-\infty}^{\infty} 2^{[(l-k)s-2|k-l|\theta]q}\right\}^{1/q} 2^{ks}\left[\int_{\mathbb{R}^d\setminus Q_{0,M}} \left|\left(D_k T_N^{-1}\right)(g)(y)\right|^p\, d\mu(y)\right]^{1/p}\\
&\le CN 2^{ks}\left[\int_{\mathbb{R}^d\setminus Q_{0,M}} \left|\left(D_k T_N^{-1}\right)(g)(y)\right|^p\, d\mu(y)\right]^{1/p}\\
&\to 0,
\end{aligned}$$

as $M \to \infty$. Thus, (2.49) holds. Therefore,

$$\left\langle f, D_k^N D_k T_N^{-1}(g)\right\rangle = \lim_{M\to\infty} \left\langle f, g_{k,M}\right\rangle. \tag{2.50}$$

Let $S = Q_{0,M} \cap \operatorname{supp}(\mu)$. For any $z \in S$, there is a cube $Q_{z,k+N}$ centered at z. Thus, $\{Q_{z,k+N}\}_{z\in S}$ is a covering of S. By the compactness of S, we can find a finite number of cubes, $\{Q_{z_i,k+N}\}_{i=1}^{\nu} \subset \{Q_{z,k+N}\}_{z\in S}$, such that $\bigcup_{i=1}^{\nu} Q_{z_i,k+N} \supset S$. We now decompose S into the union of a finite number of cubes with disjoint interior, $\{Q_j\}_{j=1}^{N_0}$, such that each Q_j for $j \in \{1, \cdots, N_0\}$ is contained in some $Q_{z_i,k+N}$ for some $i \in \{1, \cdots, \nu\}$. We then divide each Q_j into a union of cubes, $\{Q_j^i\}_{i=1}^{N_j}$, such that $\ell(Q_j^i) \sim 2^{-J}$, where $N_j \sim 2^J \ell(Q_j)$ for $j = 1, \cdots, N_0$. Now we write

$$\begin{aligned} g_{k,M}(x) &= \sum_{j=1}^{N_0} \int_{Q_j} D_k^N(x,y) \left(D_k T_N^{-1}\right)(g)(y)\, d\mu(y) \\ &= \sum_{j=1}^{N_0} \sum_{i=1}^{N_j} \int_{Q_j^i} \left[D_k^N(x,y) - D_k^N(x, y_{Q_j^i})\right] \left(D_k T_N^{-1}\right)(g)(y)\, d\mu(y) \\ &\quad + \sum_{j=1}^{N_0} \sum_{i=1}^{N_j} D_k^N(x, y_{Q_j^i}) \int_{Q_j^i} \left(D_k T_N^{-1}\right)(g)(y)\, d\mu(y) \\ &= g_{k,M}^1(x) + g_{k,M}^2(x), \end{aligned}$$

where $y_{Q_j^i}$ is any point in the cube Q_j^i. We now claim that for any fixed k and M,

$$\lim_{J\to\infty} \left\|g_{k,M}^1\right\|_{\dot{B}^s_{pq}(\mu)} = 0. \tag{2.51}$$

To prove this claim, the second difference smoothness condition will be used. Let

$$F_{k,i,j}(z,y) = \left[D_k^N(z,y) - D_k^N(z, y_{Q_j^i})\right] \chi_{Q_j^i}(y).$$

Lemmas 2.4 and 2.5 tell us that

$$\operatorname{supp} F_{k,i,j}(\cdot, y) \subset Q_{y,k-N-3}, \quad \operatorname{supp} F_{k,i,j}(z,\cdot) \subset Q_{z,k-N-3}; \tag{2.52}$$

$$\int_{\mathbb{R}^d} F_{k,i,j}(z,y)\, d\mu(z) = 0; \tag{2.53}$$

$$|F_{k,i,j}(z,y)| \le C_3 2^{-J} \ell(Q_{z_{i_0},k+N})^{-1} \frac{1}{\left(\ell(Q_{z,k+N}) + \ell(Q_{y,k+N}) + |z-y|\right)^n} \tag{2.54}$$

if $Q_j^i \subset Q_{z_{i_0},k+N}$ for some $i_0 \in \{1, \cdots, \nu\}$; and

$$\begin{aligned} &|F_{k,i,j}(z,y) - F_{k,i,j}(z',y)| \\ &\le C_3 2^{-J} \ell(Q_{z_{i_0},k+N})^{-1} \frac{|z-z'|}{\ell(Q_{x_0,k+N})} \frac{1}{\left(\ell(Q_{z,k+N}) + \ell(Q_{y,k+N}) + |z-y|\right)^n} \end{aligned} \tag{2.55}$$

if $z,\ z' \in Q_{x_0,k+N}$ for some $x_0 \in \operatorname{supp}(\mu)$ and $Q_j^i \subset Q_{z_{i_0},k+N}$ for some $i_0 \in \{1, \cdots, \nu\}$. Here C_3 depends on N. From (2.52), (2.53), (2.54), (2.55), Lemma 2.5 and its proof, it follows that for all $l,\ k \in \mathbb{Z}$ and $x,\ y \in \operatorname{supp}(\mu)$,

$$\operatorname{supp}\, \left(D_l F_{k,i,j}\right)(\cdot, y) \subset Q_{y,\min(l,k-N-1)-3}, \tag{2.56}$$

$$\operatorname{supp}\, \left(D_l F_{k,i,j}\right)(x, \cdot) \subset Q_{x,\min(l,k-N-1)-3}, \tag{2.57}$$

and for all $x \in \operatorname{supp}(\mu)$ and $y \in Q_j^i \subset Q_{z_{i_0},k+N}$ for some $i_0 \in \{1, \cdots, \nu\}$,

(2.58)

$$\begin{aligned} |\left(D_l F_{k,i,j}\right)(x,y)| &\le C_3 2^{-J} 2^{-2|l-k|\theta} \ell(Q_{z_{i_0},k+N})^{-1} \\ &\quad \times \frac{1}{\left(\ell(Q_{x,\min(l,k+N)+1}) + \ell(Q_{y,\min(l,k+N)+1}) + |x-y|\right)^n}. \end{aligned}$$

Let

$$C_4 = \max\left\{ C_3 \frac{1}{\ell(Q_{z_i,k+N})} :\ i = 1, \cdots, \nu \right\}.$$

Then C_4 depends on N, k, but not on J and l. Set

$$K(x,y) = \sum_{j=1}^{N_0} \sum_{i=1}^{N_j} (D_l F_{k,i,j})(x,y).$$

Then, by (2.57) and (2.58), we have

$$\begin{aligned}(2.59)\quad \int_{\mathbb{R}^d} |K(x,y)|\, d\mu(y) &\le CC_4 2^{-J} 2^{-2|l-k|\theta} \\ &\quad \times \sum_{j=1}^{N_0} \sum_{i=1}^{N_j} \int_{Q_j^i \cap Q_{x,\min(l,k-N-1)-3}} \frac{1}{\left(\ell(Q_{x,\min(l,k+N)+1}) + |x-y|\right)^n}\, d\mu(y) \\ &= CC_4 2^{-J} 2^{-2|l-k|\theta} \\ &\quad \times \sum_{j=1}^{N_0} \int_{Q_j \cap Q_{x,\min(l,k-N-1)-3}} \frac{1}{\left(\ell(Q_{x,\min(l,k+N)+1}) + |x-y|\right)^n}\, d\mu(y) \\ &\le CC_4 N_0 2^{-J} 2^{-2|l-k|\theta} \left\{ \int_{Q_{x,\min(l,k+N)+1}} \frac{d\mu(y)}{\ell(Q_{x,\min(l,k+N)+1})^n} \right. \\ &\quad \left. + \delta(Q_{x,\min(l,k-N-1)-3}, Q_{x,\min(l,k+N)+1}) \right\} \\ &\le C_5 2^{-J} 2^{-2|l-k|\theta},\end{aligned}$$

and, similarly, by (2.56) and (2.58),

$$(2.60)\quad \int_{\mathbb{R}^d} |K(x,y)|\, d\mu(x) \le C_5 2^{-J} 2^{-2|l-k|\theta},$$

where C_5 is independent of J and l, but, it may depend on M, N and k. Therefore, from Schur's Lemma together with (2.59) and (2.60), it follows that

$$\|D_l(g^1_{k,M})\|_{L^p(\mu)} \le C_5 2^{-J} 2^{-2|l-k|\theta} \|(D_k T_N^{-1})(g)\|_{L^p(\mu)},$$

and, from this, it further follows that

$$\begin{aligned}(2.61)\quad \left\|g^1_{k,M}\right\|_{\dot{B}^s_{pq}(\mu)} &\le C_5 2^{-J} \left\{ \sum_{l=-\infty}^{\infty} 2^{lsq} 2^{-2|l-k|\theta q} \right\}^{1/q} \left\|\left(D_k T_N^{-1}\right)(g)\right\|_{L^p(\mu)} \\ &\le CC_5 2^{-J} 2^{ks} \left\|\left(D_k T_N^{-1}\right)(g)\right\|_{L^p(\mu)} \\ &\to 0,\end{aligned}$$

as $J \to \infty$. Obviously, (2.61) implies (2.51). By (2.50) and (2.51), we have

(2.62)

$$\begin{aligned}\left\langle f, D_k^N D_k T_N^{-1}(g) \right\rangle &= \lim_{M\to\infty} \langle f, g_{k,M} \rangle \\ &= \lim_{M\to\infty} \lim_{J\to\infty} \left\langle f, g^2_{k,M} \right\rangle \\ &= \lim_{M\to\infty} \lim_{J\to\infty} \sum_{j=1}^{N_0} \sum_{i=1}^{N_j} D_k^N(f)(y_{Q_j^i}) \int_{Q_j^i} \left(D_k T_N^{-1}\right)(g)(y)\, d\mu(y).\end{aligned}$$

We now write

$$\begin{aligned}
&\sum_{i=1}^{N_j} D_k^N(f)(y_{Q_j^i}) \int_{Q_j^i} \left(D_k T_N^{-1}\right)(g)(y)\, d\mu(y)\\
&\quad = \sum_{i=1}^{N_j} \int_{Q_j^i} D_k^N(f)(y) \left(D_k T_N^{-1}\right)(g)(y)\, d\mu(y)\\
&\quad + \int_{\mathbb{R}^d} \left\{ \sum_{i=1}^{N_j} \left[D_k^N(f)(y_{Q_j^i}) - D_k^N(f)(y) \right] \chi_{Q_j^i}(y) \right\} \left(D_k T_N^{-1}\right)(g)(y)\, d\mu(y).
\end{aligned}$$

Using the second difference property of the approximation to the identity in Lemma 2.4(f), by a proof similar to that for (2.61), we can show that

$$\left\| \sum_{i=1}^{N_j} \left[D_k^N(y_{Q_j^i}, \cdot) - D_k^N(y, \cdot) \right] \chi_{Q_j^i}(\cdot) \right\|_{\dot{B}^s_{pq}(\mu)} \le C_6 2^{-J},$$

where C_6 is independent of J. From this, it follows that

$$\left| \sum_{i=1}^{N_j} \left[D_k^N(f)(y_{Q_j^i}) - D_k^N(f)(y) \right] \chi_{Q_j^i}(y) \right| \le C_6 2^{-J} \|f\|_{\left(\dot{\mathcal{B}}^s_{pq}(\mu)\right)'}$$

for all $y \in \operatorname{supp}(\mu)$. Noting that $\left(D_k T_N^{-1}\right)(g) \in L^q(\mu)$ by Theorem 2.1 and the construction of $\{Q_j^i\}$ for $j \in \{1, \cdots, N_0\}$ and $i \in \{1, \cdots, N_j\}$, by the Lebesgue dominated convergence theorem, we have

$$\lim_{J\to\infty} \int_{\mathbb{R}^d} \left\{ \sum_{i=1}^{N_j} \left[D_k^N(f)(y_{Q_j^i}) - D_k^N(f)(y) \right] \chi_{Q_j^i}(y) \right\} \left(D_k T_N^{-1}\right)(g)(y)\, d\mu(y) = 0.$$

Thus, by this fact together with (2.62), we further have

$$\begin{aligned}
\left\langle f, D_k^N D_k T_N^{-1}(g) \right\rangle &= \lim_{M\to\infty} \lim_{J\to\infty} \sum_{j=1}^{N_0} \sum_{i=1}^{N_j} \int_{Q_j^i} D_k^N(f)(y) \left(D_k T_N^{-1}\right)(g)(y)\, d\mu(y)\\
&= \int_{\mathbb{R}^d} D_k^N(f)(y) \left(D_k T_N^{-1}\right)(g)(y)\, d\mu(y)\\
&= \left\langle T_N^{-1} D_k D_k^N(f), g \right\rangle.
\end{aligned}$$

That is, (2.48) holds, and we have completed the proof of Theorem 2.2.

3. Besov spaces

It is easy to see that $D_k(x, \cdot) \in L^2(\mu)$ with compact support for all $x \in \operatorname{supp}(\mu)$ and all $k \in \mathbb{Z}$. Let $|s| < \theta$. We will show that $D_k(x, \cdot) \in \dot{\mathcal{B}}^s_{pq}(\mu)$ for all $x \in \operatorname{supp}(\mu)$ and $1 \le p,\ q \le \infty$.

Lemma 3.1. *Let θ be the same as in Definition 2.5, $|s| < \theta$ and $1 \le p,\ q \le \infty$. Let $\{D_k\}_{k=-\infty}^{\infty}$ be the same as in Theorem 2.1. Then $D_k(x, \cdot)$ and $D_k(\cdot, x)$ are in $\dot{\mathcal{B}}^s_{pq}(\mu)$ for all $x \in \operatorname{supp}(\mu)$ and all $k \in \mathbb{Z}$.*

Proof. Noting that $D_k(x,\cdot) = D_k(\cdot,x)$, we only need to verify the lemma for $D_k(x,\cdot)$. For any $k \in \mathbb{Z}$, any $x \in \operatorname{supp}(\mu)$ and $p = 1$, by Lemma 2.5, we first have

(3.1)

$$\begin{aligned}\|D_jD_k(\cdot,x)\|_{L^p(\mu)} &\le C2^{-2|j-k|\theta}\int_{Q_{x,\min(j,k)-3}} \frac{1}{(\ell(Q_{x,\min(j,k)+1})+|y-x|)^n}\,d\mu(y)\\ &\le C2^{-2|j-k|\theta}\left\{\int_{Q_{x,\min(j,k)+1}} \frac{1}{(\ell(Q_{x,\min(j,k)+1})+|y-x|)^n}\,d\mu(y)\right.\\ &\quad\left.+\int_{Q_{x,\min(j,k)-3}\setminus Q_{x,\min(j,k)+1}}\cdots\right\}\\ &\le C2^{-2|j-k|\theta}\delta(Q_{x,\min(j,k)+1},Q_{x,\min(j,k)+1})\\ &\le C2^{-2|j-k|\theta};\end{aligned}$$

if $p=\infty$, it is obvious that we have

$$\|D_jD_k(\cdot,x)\|_{L^p(\mu)} \le C2^{-2|j-k|\theta}\frac{1}{\ell(Q_{x,k+1})^n}; \tag{3.2}$$

and if $1<p<\infty$, then

$$\begin{aligned}\|D_jD_k(\cdot,x)\|_{L^p(\mu)} &\le \|D_jD_k(\cdot,x)\|_{L^1(\mu)}^{1/p}\|D_jD_k(\cdot,x)\|_{L^\infty(\mu)}^{1-1/p}\\ &\le C2^{-2|j-k|\theta}\frac{1}{\ell(Q_{x,k+1})^{(1-1/p)n}}.\end{aligned} \tag{3.3}$$

Combining the estimates (3.1), (3.2) and (3.3) yields that

$$\begin{aligned}\|D_k(x,\cdot)\|_{\dot{B}^s_{pq}(\mu)} &= \left\{\sum_{j=-\infty}^{\infty} 2^{jsq}\|D_jD_k(\cdot,x)\|^q_{L^p(\mu)}\right\}^{1/q}\\ &\le C\frac{1}{\ell(Q_{x,k+1})^{(1-1/p)n}}\left\{\sum_{j=-\infty}^{\infty}2^{jsq-2\theta|j-k|q}\right\}^{1/q}\\ &\le C2^{ks}\frac{1}{\ell(Q_{x,k+1})^{(1-1/p)n}},\end{aligned}$$

where C is independent of k and x. This proves the lemma. □

Remark 3.1. More generally, we can verify that if $f \in C^1(\mathbb{R}^d)$ with compact support and

$$\int_{\mathbb{R}^d} f(x)\,d\mu(x) = 0,$$

then $f \in \dot{\mathcal{B}}^s_{pq}(\mu)$ for $|s|<\theta$ with θ the same as in Definition 2.5 and $1 \le p,\ q \le \infty$. We leave the details to the reader.

We can now introduce the Besov spaces $\dot{B}^s_{pq}(\mu)$.

Definition 3.1. Let θ be the same as in Definition 2.5 and let $|s|<\theta$. Let p' and q' be the conjugate index of p and q, respectively, and let $\{D_k\}_{k=-\infty}^{\infty}$ be as in

Theorem 2.1. For $1 \le p,\ q \le \infty$, we define

$$\dot{B}^s_{pq}(\mu) = \left\{ f \in \left(\dot{\mathcal{B}}^{-s}_{p',q'}(\mu) \right)' : \ \|f\|_{\dot{B}^s_{pq}(\mu)} < \infty \right\},$$

where

$$\|f\|_{\dot{B}^s_{pq}(\mu)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} \|D_k f\|^q_{L^p(\mu)} \right\}^{1/q}.$$

Based on Lemma 3.1 and Theorem 2.2, for all $f \in \left(\dot{\mathcal{B}}^{-s}_{p',q'}(\mu) \right)'$ with $1 \le p,\ q \le \infty$, we have

$$E_j f(x) = \sum_{k=-\infty}^{\infty} E_j D^N_k D_k T_N^{-1}(f)(x),$$

where $\{P_k\}_{k\in\mathbb{Z}}$ is an approximation to the identity as in Definition 2.9, $E_k = P_k - P_{k-1}$ for $k \in \mathbb{Z}$, and $N \in \mathbb{N}$ is large enough such that (2.45) holds. The above equality and the same proof of Proposition 2.1 show that spaces $\dot{B}^s_{pq}(\mu)$ are independent of the choice of approximations to the identity as in Definition 2.9. We leave these details to the reader.

It is well known that the space of Schwartz functions is dense in Besov spaces on $\mathbb{R}^d$. The following result shows that our space of test functions, $\dot{\mathcal{B}}^s_{p,q}(\mu)$, is also dense in the Besov space $\dot{B}^s_{pq}(\mu)$.

Proposition 3.1. *Let θ be the same as in Definition 2.5, $|s| < \theta$ and $1 \le p,\ q \le \infty$. Then*

$$\overline{\dot{\mathcal{B}}^s_{pq}(\mu)} = \dot{B}^s_{pq}(\mu), \tag{3.4}$$

where $\overline{\dot{\mathcal{B}}^s_{pq}(\mu)}$ is the closure of $\dot{\mathcal{B}}^s_{pq}(\mu)$ with respect to the norm $\|\cdot\|_{\dot{B}^s_{pq}(\mu)}$.

Proof. We first show $\overline{\dot{\mathcal{B}}^s_{pq}(\mu)} \subset \dot{B}^s_{pq}(\mu)$. To do this, we claim that if $1 \le p,\ q \le \infty$ and $f \in \dot{\mathcal{B}}^s_{pq}(\mu)$, then $f \in \left(\dot{\mathcal{B}}^{-s}_{p',q'}(\mu) \right)'$ and

$$\|f\|_{\left(\dot{\mathcal{B}}^{-s}_{p',q'}(\mu)\right)'} \le C \|f\|_{\dot{B}^s_{pq}(\mu)}. \tag{3.5}$$

Moreover, let $\{f_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence of $\dot{\mathcal{B}}^s_{pq}(\mu)$ according to the norm $\|\cdot\|_{\dot{B}^s_{pq}(\mu)}$. Then there is an $f \in \left(\dot{\mathcal{B}}^{-s}_{p',q'}(\mu) \right)'$ such that $\|f\|_{\dot{B}^s_{pq}(\mu)} < \infty$ and $f_k \to f$ in $\dot{B}^s_{pq}(\mu)$ as $k \to \infty$.

To show the above claim, let $f \in \dot{\mathcal{B}}^s_{pq}(\mu)$ and $g \in \dot{\mathcal{B}}^{-s}_{p',q'}(\mu)$. Let $\{D_k\}_{k\in\mathbb{Z}}$ be the same as in Theorem 2.1 and recall $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}$. It is easy to see that D^N_k for $k \in \mathbb{Z}$ has the same properties as those of D_k for $k \in \mathbb{Z}$, deduced from Lemma 2.4, with a constant depending on N, namely CN, if C is the constant appearing in the properties satisfied by D_k for $k \in \mathbb{Z}$; see Lemma 2.4.

Noting $\left(D_k^N\right)^* = D_k^N$, by (2.37), the Hölder inequality, Theorem 2.1 and Proposition 2.1, we obtain

$$\begin{aligned}
|f(g)| &= |\langle f, g\rangle| \quad \Big(\text{in the sense of } \left(L^2(\mu)\right)' = L^2(\mu)\Big) \\
&= \left| \int_{\mathbb{R}^d} \sum_{k=-\infty}^{\infty} D_k^N D_k T_N^{-1}(f) g \, d\mu \right| \\
&= \left| \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^d} D_k T_N^{-1}(f) D_k^N(g) \, d\mu \right| \\
&\le \sum_{k=-\infty}^{\infty} \left\| D_k T_N^{-1}(f) \right\|_{L^p(\mu)} \left\| D_k^N(g) \right\|_{L^{p'}(\mu)} \\
&\le \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} \left\| D_k T_N^{-1}(f) \right\|_{L^p(\mu)}^q \right\}^{1/q} \left\{ \sum_{k=-\infty}^{\infty} 2^{-ksq'} \left\| D_k^N(g) \right\|_{L^{p'}(\mu)}^{q'} \right\}^{1/q'} \\
&\le C_N \left\| T_N^{-1}(f) \right\|_{\dot{B}^s_{pq}(\mu)} \|g\|_{\dot{B}^{-s}_{p',q'}(\mu)} \\
&\le C_N \|f\|_{\dot{B}^s_{pq}(\mu)} \|g\|_{\dot{B}^{-s}_{p',q'}(\mu)},
\end{aligned}$$

where $C_N > 0$ depends on N. Thus, $f \in \left(\dot{\mathcal{B}}^{-s}_{p',q'}(\mu)\right)'$ and

$$\|f\|_{\left(\dot{\mathcal{B}}^{-s}_{p',q'}(\mu)\right)'} \le C \|f\|_{\dot{B}^s_{pq}(\mu)}.$$

That is, (3.5) holds.

Now let $\{f_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence of $\dot{\mathcal{B}}^s_{pq}(\mu)$ according to the norm $\|\cdot\|_{\dot{B}^s_{pq}(\mu)}$. Then, by (3.5), it is also a Cauchy sequence according to the norm $\|\cdot\|_{\left(\dot{\mathcal{B}}^{-s}_{p',q'}(\mu)\right)'}$. Since $\left(\dot{\mathcal{B}}^{-s}_{p',q'}(\mu)\right)'$ is a Banach space (see [33]), then there is an $f \in \left(\dot{\mathcal{B}}^{-s}_{p',q'}(\mu)\right)'$ such that $f_k \to f$ in $\left(\dot{\mathcal{B}}^{-s}_{p',q'}(\mu)\right)'$ as $k \to \infty$. We still need to verify that $\|f\|_{\dot{B}^s_{pq}(\mu)} < \infty$. From Lemma 3.1 and

$$|D_k(f_n - f)(x)| \le \|D_k(x,\cdot)\|_{\dot{B}^s_{pq}(\mu)} \|f_n - f\|_{\left(\dot{\mathcal{B}}^{-s}_{p',q'}(\mu)\right)'},$$

it follows that for all $x \in \operatorname{supp}(\mu)$ and all $k \in \mathbb{Z}$,

$$\lim_{n\to\infty} D_k f_n(x) = D_k f(x). \tag{3.6}$$

Thus, the fact that $\|f_n\|_{\dot{B}^s_{pq}(\mu)} \le C$ with C independent of n, Definition 3.1, the Fatou lemma and (3.6) tell us that

$$\|f\|_{\dot{B}^s_{pq}(\mu)} \le C,$$

which shows $f \in \dot{B}^s_{pq}(\mu)$ and $f_k \to f$ in $\dot{B}^s_{pq}(\mu)$ as $k \to \infty$.

We now prove the other direction: $\dot{B}^s_{pq}(\mu) \subset \overline{\dot{\mathcal{B}}^s_{pq}(\mu)}$. This fact comes from Theorem 2.2 and its proof. More precisely, if $f \in \dot{B}^s_{pq}(\mu)$, then by Theorem 2.2 and its proof, we can write (2.46) as

$$f = \sum_{k\in\mathbb{Z}} D_k D_k^N T_N^{-1}(f), \tag{3.7}$$

where the series converges in the norm of $\dot{B}^s_{pq}(\mu)$.

As in the proof of Theorem 2.2, if we define $g_{k,M}(x)$ by

$$g_{k,M}(x) = \int_{Q_{0,M}} D_k(x,y)\left(D_k^N T_N^{-1}\right)(f)(y)\,d\mu(y),$$

then $g_{k,M}(x)$ belongs to $\dot{\mathcal{B}}^s_{pq}(\mu)$ by Remark 3.1, and using (3.7), we can further verify f can be approximated by a finite sum of $g_{k,M}(x)$. We leave the details to the reader. This shows $\dot{B}^s_{pq}(\mu) \subset \overline{\dot{\mathcal{B}}^s_{pq}(\mu)}$, and we have completed the proof of Proposition 3.1.

We remark that as a consequence, Proposition 3.1 indicates that $\dot{B}^s_{pq}(\mu)$ is a Banach space for $1 \le p,\ q \le \infty$.

We now establish the boundedness of the Riesz operators defined via the approximation to the identity in the spaces $\dot{B}^s_{pq}(\mu)$, and then we show that the spaces $\dot{B}^s_{pq}(\mu)$ have the lifting property by using these Riesz potential operators.

Definition 3.2. Let $\{D_k\}_{k\in\mathbb{Z}}$ be the same as in Theorem 2.1. For $\alpha \in \mathbb{R}$, $f \in L^2(\mu)$ and all $x \in \operatorname{supp}(\mu)$, we define the Riesz potential operator I_α by

$$I_\alpha f(x) = \sum_{k=-\infty}^{\infty} 2^{-k\alpha} D_k f(x).$$

Theorem 3.1. *Let θ be the same as in Definition* 2.5*, let $|s| < \theta$ and let $|s+\alpha| < \theta$. Then I_α is bounded from $\dot{B}^s_{pq}(\mu)$ to $\dot{B}^{s+\alpha}_{pq}(\mu)$ for $1 \le p,\ q \le \infty$, namely, there is a constant $C > 0$ such that for all $f \in \dot{B}^s_{pq}(\mu)$ with $s,\ p$ and q as above,*

$$\|I_\alpha f\|_{\dot{B}^{s+\alpha}_{pq}(\mu)} \le C\|f\|_{\dot{B}^s_{pq}(\mu)}.$$

Proof. Let $\{D_k\}_{k\in\mathbb{Z}}$ be the same as in Theorem 2.1. From Theorem 2.2 and the Minkowski inequality, it follows that

(3.8)

$$\begin{aligned}\|I_\alpha f\|_{\dot{B}^{s+\alpha}_{pq}(\mu)} &= \left\{\sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)q}\|D_j I_\alpha f\|^q_{L^p(\mu)}\right\}^{1/q}\\ &\le \left\{\sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)q}\left[\sum_{k=-\infty}^{\infty}\|D_j I_\alpha D_k^N D_k T_N^{-1} f\|_{L^p(\mu)}\right]^q\right\}^{1/q}\\ &\le \left\{\sum_{j=-\infty}^{\infty} 2^{j(s+\alpha)q}\left[\sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} 2^{-i\alpha}\|D_j D_i D_k^N D_k T_N^{-1} f\|_{L^p(\mu)}\right]^q\right\}^{1/q},\end{aligned}$$

where we assume that N satisfies (2.31).

Since $|s| < \theta$ and $|s+\alpha| < \theta$, we can choose $\nu \in (0, 1/2)$ such that $|s+\alpha| < 2\nu\theta$, $|s| < 2\nu\theta$ and $|s| < 2(1-\nu)\theta$. Similar to (2.41) and (2.42), by Lemma 2.5, we have

$$\|D_j D_i D_k^N D_k T_N^{-1} f\|_{L^p(\mu)} \le C 2^{-2\theta|j-i|}\|D_k T_N^{-1} f\|_{L^p(\mu)} \tag{3.9}$$

and

$$\|D_j D_i D_k^N D_k T_N^{-1} f\|_{L^p(\mu)} \le C 2^{-2\theta|i-k|}\|D_k T_N^{-1} f\|_{L^p(\mu)}. \tag{3.10}$$

The geometric means of (3.9) and (3.10) tells us that

$$\|D_j D_i D_k^N D_k T_N^{-1} f\|_{L^p(\mu)} \le C 2^{-2\theta\nu|j-i|} 2^{-2\theta(1-\nu)|i-k|}\|D_k T_N^{-1} f\|_{L^p(\mu)}. \tag{3.11}$$

Inserting (3.11) into (3.8), the Hölder inequality and Theorem 2.1 yield that

$$\begin{aligned}
\|I_\alpha f\|_{\dot{B}^{s+\alpha}_{pq}(\mu)} &\le C\left\{\sum_{j=-\infty}^{\infty}\left[\sum_{i=-\infty}^{\infty} 2^{(j-i)(s+\alpha)-2\theta\nu|j-i|}\right.\right.\\
&\qquad\left.\left.\times\sum_{k=-\infty}^{\infty} 2^{is}2^{-2\theta(1-\nu)|i-k|}\|D_kT_N^{-1}f\|_{L^p(\mu)}\right]^q\right\}^{1/q}\\
&\le C\left\{\sum_{j=-\infty}^{\infty}\left[\left(\sum_{i=-\infty}^{\infty} 2^{(j-i)(s+\alpha)-2\theta\nu|j-i|}\right.\right.\right.\\
&\qquad\left.\times\left[\sum_{k=-\infty}^{\infty} 2^{is}2^{-2\theta(1-\nu)|i-k|}\|D_kT_N^{-1}f\|_{L^p(\mu)}\right]^q\right)^{1/q}\\
&\qquad\left.\left.\times\left(\sum_{i=-\infty}^{\infty} 2^{(j-i)(s+\alpha)-2\theta\nu|j-i|}\right)^{1/q'}\right]^q\right\}^{1/q}\\
&\le C\left\{\sum_{i=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} 2^{(i-k)s-2\theta(1-\nu)|i-k|}2^{ks}\|D_kT_N^{-1}f\|_{L^p(\mu)}\right]^q\right\}^{1/q}\\
&\le C\left\{\sum_{i=-\infty}^{\infty}\left[\left(\sum_{k=-\infty}^{\infty} 2^{(i-k)s-2\theta(1-\nu)|i-k|}2^{ksq}\|D_kT_N^{-1}f\|^q_{L^p(\mu)}\right)^{1/q}\right.\right.\\
&\qquad\left.\left.\times\left(\sum_{k=-\infty}^{\infty} 2^{(i-k)s-2\theta(1-\nu)|i-k|}\right)^{1/q'}\right]^q\right\}^{1/q}\\
&\le C\left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\|D_kT_N^{-1}f\|^q_{L^p(\mu)}\right\}^{1/q}\\
&= C\|T_N^{-1}f\|_{\dot{B}^s_{pq}(\mu)}\\
&\le C\|f\|_{\dot{B}^s_{pq}(\mu)},
\end{aligned}$$

where $1/q+1/q'=1$. This proves that I_α is bounded from $\dot{B}^s_{pq}(\mu)$ to $\dot{B}^{s+\alpha}_{pq}(\mu)$ and we have completed the proof of Theorem 3.1.

We now establish the converse of Theorem 3.1. To this end, we will first show that when α is very small, the composition operator $I_\alpha I_{-\alpha}$ is invertible in the spaces $\dot{B}^s_{pq}(\mu)$. To do so, for any given $N_1\in\mathbb{N}$, we decompose $I-I_\alpha I_{-\alpha}$ into

$$\begin{aligned}
I-I_\alpha I_{-\alpha} &= \sum_{i=-\infty}^{\infty}\sum_{|m|\le N_1}(1-2^{m\alpha})D_iD_{i+m}\\
&\quad+\sum_{i=-\infty}^{\infty}\sum_{|m|>N_1}(1-2^{m\alpha})D_iD_{i+m}\\
&= L^1_{N_1}+L^2_{N_1}.
\end{aligned}$$

We will show that if N_1 is large enough and if α is small enough, then the operator norms of $L^i_{N_1}$ in the spaces $\dot{B}^s_{pq}(\mu)$ will be very small for $i = 1,\ 2$. Thus, $I_\alpha I_{-\alpha}$ is invertible in the spaces $\dot{B}^s_{pq}(\mu)$.

Theorem 3.2. *Let θ be the same as in Definition 2.5, let $|s| < \theta$ and let $|s-\alpha| < \theta$. Then for any $\nu \in (0, 1/2)$ such that $|s| < 2\nu\theta$ and $|s-\alpha| < 2\nu\theta$,*

$$\|L^1_{N_1}\|_{\dot{B}^s_{pq}(\mu)\to\dot{B}^s_{pq}(\mu)} \le C_5 \sum_{|m|\le N_1} |1-2^{m\alpha}|2^{-2\theta\nu|m|-ms} \tag{3.12}$$

and

$$\|L^2_{N_1}\|_{\dot{B}^s_{pq}(\mu)\to\dot{B}^s_{pq}(\mu)} \le C_5 \sum_{|m|> N_1} |1-2^{m\alpha}|2^{-2\theta\nu|m|-ms} \tag{3.13}$$

for $1 \le p,\ q \le \infty$. Here C_5 is independent of N_1 and α.

Proof. We only show (3.12). The proof of (3.13) is similar.

To show (3.12), let $\{D_k\}_{k\in\mathbb{Z}}$ be the same as in Theorem 2.1. By Theorem 2.2, for any $j \in \mathbb{Z}$, we can write

$$D_j L^1_{N_1} f(x) = \sum_{i=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{|m|\le N_1} (1-2^{m\alpha}) D_j D_i D_{i+m} D^N_k D_k T^{-1}_N f(x), \tag{3.14}$$

where N is as in (2.31).

Let ν be the same as in Theorem 3.2. Similar to (3.9) and (3.10), by Lemma 2.5, we have

$$\begin{aligned} &\|D_j D_i D_{i+m} D^N_k D_k T^{-1}_N f\|_{L^p(\mu)} \\ &\le CN 2^{-2\theta[|j-i|+|i+m-k|]} \|D_k T^{-1}_N f\|_{L^p(\mu)} \end{aligned} \tag{3.15}$$

and

$$\|D_j D_i D_{i+m} D^N_k D_k T^{-1}_N f\|_{L^p(\mu)} \le CN 2^{-2\theta|m|} \|D_k T^{-1}_N f\|_{L^p(\mu)}. \tag{3.16}$$

The geometric means of (3.15) and (3.16) implies that

$$\begin{aligned} &\|D_j D_i D_{i+m} D^N_k D_k T^{-1}_N f\|_{L^p(\mu)} \\ &\le CN 2^{-2\theta(1-\nu)[|j-i|+|i+m-k|]} 2^{-2\theta\nu|m|} \|D_k T^{-1}_N f\|_{L^p(\mu)}. \end{aligned} \tag{3.17}$$

The formula (3.14), the estimate (3.17), the Minkowski inequality, the Hölder inequality and Theorem 2.1 tell us that

$$\begin{aligned}
&\|L^1_{N_1} f\|_{\dot{B}^s_{pq}(\mu)} \\
&\quad = \left\{ \sum_{j=-\infty}^{\infty} 2^{jsq} \|D_j L^1_{N_1} f\|^q_{L^p(\mu)} \right\}^{1/q} \\
&\quad \le C \sum_{|m|\le N_1} |1-2^{m\alpha}| 2^{-2\theta\nu|m|} \left\{ \sum_{j=-\infty}^{\infty} \left[\sum_{i=-\infty}^{\infty} 2^{(j-i)s-2\theta(1-\nu)|j-i|} \right.\right. \\
&\qquad \left.\left. \times \sum_{k=-\infty}^{\infty} 2^{is} 2^{-2\theta(1-\nu)|i+m-k|]} \|D_k T_N^{-1} f\|_{L^p(\mu)} \right]^q \right\}^{1/q} \\
&\quad \le C \sum_{|m|\le N_1} |1-2^{m\alpha}| 2^{-2\theta\nu|m|-ms} \\
&\qquad \times \left\{ \sum_{i=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} 2^{(i+m-k)s-2\theta(1-\nu)|i+m-k|]} 2^{ks} \|D_k T_N^{-1} f\|_{L^p(\mu)} \right]^q \right\}^{1/q} \\
&\quad \le C \sum_{|m|\le N_1} |1-2^{m\alpha}| 2^{-2\theta\nu|m|-ms} \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} \|D_k T_N^{-1} f\|^q_{L^p(\mu)} \right\}^{1/q} \\
&\quad \le C \sum_{|m|\le N_1} |1-2^{m\alpha}| 2^{-2\theta\nu|m|-ms} \|T_N^{-1} f\|_{\dot{B}^s_{pq}(\mu)} \\
&\quad \le C \sum_{|m|\le N_1} |1-2^{m\alpha}| 2^{-2\theta\nu|m|-ms} \|f\|_{\dot{B}^s_{pq}(\mu)}.
\end{aligned}$$

That is, (3.12) holds and we have finished the proof of Theorem 3.2.

From Theorem 3.2, it is easy to deduce the following result.

Corollary 3.1. *Let θ be the same as in Definition 2.5, let $|s| < \theta$ and let $|s-\alpha| < \theta$. Then there is $\alpha_0(s) > 0$ such that if $|\alpha| < \alpha_0(s)$ and $N_1 \in \mathbb{N}$ is large enough,*

$$C_5 \left\{ \sum_{|m|\le N_1} |1-2^{m\alpha}| 2^{-2\theta\nu|m|-ms} + \sum_{|m|>N_1} |1-2^{m\alpha}| 2^{-2\theta\nu|m|-ms} \right\} < 1,$$

where $\nu \in (0, 1/2)$ satisfies that $|s| < 2\nu\theta$ and $|s-\alpha| < 2\nu\theta$. Thus, if $1 \le p,\ q \le \infty$ and $|\alpha| < \alpha_0(s)$, then $(I_\alpha I_{-\alpha})^{-1}$ exists in $\dot{B}^s_{pq}(\mu)$ and

$$\|(I_\alpha I_{-\alpha})^{-1}\|_{\dot{B}^s_{pq}(\mu)\to\dot{B}^s_{pq}(\mu)} \le C.$$

If we change the order of I_α and $I_{-\alpha}$, we have a similar result which is a simple corollary of Corollary 3.1 above.

Corollary 3.2. *Let θ be the same as in Definition 2.5, let $|s| < \theta$ and let $|s+\alpha| < \theta$. Then there is $\alpha_0(s) > 0$ such that if $|\alpha| < \alpha_0(s)$ and $N_1 \in \mathbb{N}$ is large enough,*

$$C_5 \left\{ \sum_{|m|\le N_1} |1-2^{-m\alpha}| 2^{-2\theta\nu|m|-ms} + \sum_{|m|>N_1} |1-2^{-m\alpha}| 2^{-2\theta\nu|m|-ms} \right\} < 1,$$

where $\nu \in (0, 1/2)$ *satisfies that* $|s| < 2\nu\theta$ *and* $|s+\alpha| < 2\nu\theta$*. Thus, if* $1 \le p,\ q \le \infty$ *and* $|\alpha| < \alpha_0(s)$*, then* $(I_{-\alpha} I_\alpha)^{-1}$ *exists in* $\dot{B}^s_{pq}(\mu)$ *and*

$$\|(I_{-\alpha} I_\alpha)^{-1}\|_{\dot{B}^s_{pq}(\mu) \to \dot{B}^s_{pq}(\mu)} \le C.$$

Theorem 3.1 and Corollary 3.2 imply the following lifting theorem for the spaces $\dot{B}^s_{pq}(\mu)$.

Theorem 3.3. *Let* θ *be the same as in Definition* 2.5*,* $|s| < \theta$ *and* $|s+\alpha| < \theta$*. Let* $\alpha_0(s)$ *be the same as in Corollary* 3.2 *and* $|\alpha| < \alpha_0(s)$*. Then, if* $1 \le p,\ q \le \infty$*, there is a constant* $C_6 > 0$ *such that for all* $f \in \dot{B}^s_{pq}(\mu)$*,*

$$C_6^{-1} \|f\|_{\dot{B}^s_{pq}(\mu)} \le \|I_\alpha f\|_{\dot{B}^{s+\alpha}_{pq}(\mu)} \le C_6 \|f\|_{\dot{B}^s_{pq}(\mu)}.$$

Proof. To show the theorem, we only need to verify its left-hand inequality. In fact, by Corollary 3.2, we have

$$\|f\|_{\dot{B}^s_{pq}(\mu)} = \|(I_{-\alpha} I_\alpha)^{-1} I_{-\alpha} I_\alpha\|_{\dot{B}^s_{pq}(\mu)} \le C \|I_{-\alpha} I_\alpha\|_{\dot{B}^s_{pq}(\mu)} \le C \|I_\alpha f\|_{\dot{B}^{s+\alpha}_{pq}(\mu)}.$$

We have completed the proof of Theorem 3.3.

Finally, in this section, we study the dual spaces of the spaces $\dot{B}^s_{pq}(\mu)$. To begin with, we establish the following lemma.

Lemma 3.2. *Let* θ *be the same as in Definition* 2.5*, let* $|s| < \theta$*, and let* $\{D_k\}_{k\in\mathbb{Z}}$ *be the same as in Theorem* 2.1*. Suppose that* $\{g_k\}_{k\in\mathbb{Z}}$ *is a sequence of functions on* $\mathbb{R}^d$*. If* $1 \le p,\ q < \infty$ *and*

$$\left\{ \sum_{k\in\mathbb{Z}} 2^{ksq} \|g_k\|^q_{L^p(\mu)} \right\}^{1/q} < \infty,$$

then $g(x) = \sum_{k\in\mathbb{Z}} D_k g_k(x) \in \dot{B}^s_{pq}(\mu)$ *and*

$$\|g\|_{\dot{B}^s_{pq}(\mu)} \le C \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} \|g_k\|^q_{L^p(\mu)} \right\}^{1/q},$$

where C *is a positive constant.*

Proof. For $L_1,\ L_2 \in \mathbb{Z}$ and $L_1 < L_2$, we define

$$g^{L_2}_{L_1}(x) = \sum_{k=L_1}^{L_2} D_k g_k(x).$$

Then for $f \in \dot{\mathcal{B}}^{-s}_{p',q'}(\mu)$, noting that $D_k(x,y) = D_k(y,x)$ and by the Hölder inequality, we have

$$\begin{aligned}
\left|\left\langle g_{L_1}^{L_2}, f\right\rangle\right| &= \left|\sum_{k=L_1}^{L_2} \langle D_k g_k, f\rangle\right| \\
&\le \sum_{k=L_1}^{L_2} |\langle g_k, D_k f\rangle| \\
&\le \sum_{k=L_1}^{L_2} \|g_k\|_{L^p(\mu)} \|D_k f\|_{L^{p'}(\mu)} \\
&\le \left\{\sum_{k=L_1}^{L_2} 2^{ksq} \|g_k\|_{L^p(\mu)}^q\right\}^{1/q} \left\{\sum_{k=L_1}^{L_2} 2^{-ksq'} \|D_k f\|_{L^{p'}(\mu)}^{q'}\right\}^{1/q'} \\
&\le \left\{\sum_{k=L_1}^{L_2} 2^{ksq} \|g_k\|_{L^p(\mu)}^q\right\}^{1/q} \|f\|_{\dot{\mathcal{B}}^{-s}_{p',q'}(\mu)}.
\end{aligned}$$

Thus, $g_{L_1}^{L_2} \in (\dot{\mathcal{B}}^{-s}_{p',q'}(\mu))'$ and

$$\left\|g_{L_1}^{L_2}\right\|_{\left(\dot{\mathcal{B}}^{-s}_{p',q'}(\mu)\right)'} \le \left\{\sum_{k=L_1}^{L_2} 2^{ksq} \|g_k\|_{L^p(\mu)}^q\right\}^{1/q}.$$

From this, it follows that $g \in (\dot{\mathcal{B}}^{-s}_{p',q'}(\mu))'$, and Lemma 2.5 and the Hölder inequality now tell us that

$$\begin{aligned}
\|g\|_{\dot{B}^s_{pq}(\mu)} &= \left\{\sum_{j=-\infty}^{\infty} 2^{jsq} \|D_j g\|_{L^p(\mu)}^q\right\}^{1/q} \\
&\le \left\{\sum_{j=-\infty}^{\infty} 2^{jsq} \left[\sum_{k=-\infty}^{\infty} \|D_j D_k g_k\|_{L^p(\mu)}\right]^q\right\}^{1/q} \\
&\le C\left\{\sum_{j=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} 2^{(j-k)s-2|j-k|\theta} 2^{ks} \|g_k\|_{L^p(\mu)}\right]^q\right\}^{1/q} \\
&\le C\left\{\sum_{j=-\infty}^{\infty} \left[\left(\sum_{k=-\infty}^{\infty} 2^{(j-k)s-2|j-k|\theta} 2^{ksq} \|g_k\|_{L^p(\mu)}^q\right)^{1/q}\right.\right. \\
&\qquad \left.\left.\times \left(\sum_{k=-\infty}^{\infty} 2^{(j-k)s-2|j-k|\theta}\right)^{1/q'}\right]^q\right\}^{1/q} \\
&\le C\left\{\sum_{k=-\infty}^{\infty} \left[\sum_{j=-\infty}^{\infty} 2^{(j-k)s-2|j-k|\theta}\right] 2^{ksq} \|g_k\|_{L^p(\mu)}^q\right\}^{1/q} \\
&\le C\left\{\sum_{k=-\infty}^{\infty} 2^{ksq} \|g_k\|_{L^p(\mu)}^q\right\}^{1/q}.
\end{aligned}$$

That is, $g \in \dot{B}^s_{pq}(\mu)$ and we have finished the proof of Lemma 3.2.

We now can establish the dual theorem for the spaces $\dot{B}^s_{pq}(\mu)$.

Theorem 3.4. *Let θ be the same as in Definition 2.5 and let $|s| < \theta$. If $1 \le p,\ q \le \infty$ and $g \in \dot{B}^s_{pq}(\mu)$, then*

$$\mathcal{L}_g(f) = \langle g, f\rangle$$

defines a linear functional on $\dot{\mathcal{B}}^{-s}_{p',q'}(\mu)$ and

$$\|\mathcal{L}_g\|_{\left(\dot{\mathcal{B}}^{-s}_{p',q'}(\mu)\right)'} \le C\|g\|_{\dot{B}^s_{pq}(\mu)}. \tag{3.18}$$

Conversely, if $1 < p,\ q < \infty$ and $\mathcal{L}$ is a linear functional on $\dot{B}^s_{pq}(\mu)$, then there exists a unique $g \in \dot{B}^{-s}_{p',q'}(\mu)$ such that

$$\mathcal{L}(f) = \langle g, f\rangle$$

on $\dot{\mathcal{B}}^s_{pq}(\mu)$ and

$$\|g\|_{\dot{B}^{-s}_{p',q'}(\mu)} \le C\|\mathcal{L}\|_{\left(\dot{B}^s_{pq}(\mu)\right)'}. \tag{3.19}$$

Proof. The estimate (3.18) is just (3.5) in Proposition 3.1.

Conversely, suppose that $\mathcal{L}$ is a linear functional on $\dot{B}^s_{pq}(\mu)$. By Proposition 3.1, it is easy to see that $\mathcal{L}$ is also a linear functional on $\dot{\mathcal{B}}^s_{pq}(\mu)$, and therefore, for all $f \in \dot{\mathcal{B}}^s_{pq}(\mu)$,

$$|\mathcal{L}(f)| \le \|\mathcal{L}\|_{\left(\dot{B}^s_{pq}(\mu)\right)'}\|f\|_{\dot{B}^s_{pq}(\mu)}.$$

Let $\{D_k\}_{k\in\mathbb{Z}}$ be the same as in Theorem 2.1. If $f \in \dot{\mathcal{B}}^s_{pq}(\mu)$, then the sequence $\{D_k f\}_{k\in\mathbb{Z}}$ is in the sequence space

$$l^s_q(L^p) = \left\{ \{f_k\}_{k\in\mathbb{Z}} :\ \|\{f_k\}_{k\in\mathbb{Z}}\|_{l^s_q(L^p)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq}\|f_k\|^q_{L^p(\mu)} \right\}^{1/q} < \infty \right\}.$$

Define $\widetilde{\mathcal{L}}$ on a subset of the sequence space $l^s_q(L^p)$ by

$$\widetilde{\mathcal{L}}\left[\{D_k f\}_{k\in\mathbb{Z}}\right] = \mathcal{L}(f).$$

Then, if $f \in \dot{\mathcal{B}}^s_{pq}(\mu)$, we have

$$\begin{aligned}\left|\widetilde{\mathcal{L}}\left[\{D_k f\}_{k\in\mathbb{Z}}\right]\right| &= |\mathcal{L}(f)| \le \|\mathcal{L}\|_{\left(\dot{B}^s_{pq}(\mu)\right)'}\|f\|_{\dot{B}^s_{pq}(\mu)}\\ &= \|\mathcal{L}\|_{\left(\dot{B}^s_{pq}(\mu)\right)'}\,\|\{D_k f\}_{k\in\mathbb{Z}}\|_{l^s_q(L^p)}.\end{aligned}$$

Thus, $\widetilde{\mathcal{L}}$ is bounded on this subset. The Hahn-Banach theorem tells us that $\widetilde{\mathcal{L}}$ can be extended to a functional on $l^s_q(L^p)$. Since it is well known that $\left(l^s_q(L^p)\right)' = l^{-s}_{q'}(L^{p'})$ for $1 \le p,\ q < \infty$ (see [30]), there exists a unique sequence $\{g_k\}_{k\in\mathbb{Z}} \in l^{-s}_{q'}(L^{p'})$ such that

$$\|\{g_k\}_{k\in\mathbb{Z}}\|_{l^{-s}_{q'}(L^{p'})} \le C\|\widetilde{\mathcal{L}}\|_{\left(l^s_q(L^p)\right)'} \le C\|\mathcal{L}\|_{\left(\dot{B}^s_{pq}(\mu)\right)'}$$

and

$$\widetilde{\mathcal{L}}\left[\{f_k\}_{k\in\mathbb{Z}}\right] = \sum_{k=-\infty}^{\infty} \langle g_k, f_k\rangle$$

for all $\{f_k\}_{k\in\mathbb{Z}} \in l_q^s(L^p)$. Thus, if $f \in \dot{\mathcal{B}}^s_{pq}(\mu)$, then Lemma 3.2 yields that

$$\mathcal{L}(f) = \widetilde{\mathcal{L}}\left(\{D_k f\}_{k\in\mathbb{Z}}\right) = \sum_{k=-\infty}^{\infty} \langle g_k, D_k(f)\rangle$$
$$= \sum_{k=-\infty}^{\infty} \langle D_k(g_k), f\rangle = \left\langle \sum_{k=-\infty}^{\infty} D_k(g_k), f \right\rangle,$$

since $D_k^* = D_k$. Let

$$g = \sum_{k=-\infty}^{\infty} D_k(g_k).$$

Then Lemma 3.2 tells us that $g \in \dot{B}^{-s}_{p',q'}(\mu)$ and

$$\|g\|_{\dot{B}^{-s}_{p',q'}(\mu)} \le C \left\|\{g_k\}_{k\in\mathbb{Z}}\right\|_{l^{-s}_{q'}(L^{p'})} \le C\|\mathcal{L}\|_{\left(\dot{\mathcal{B}}^s_{pq}(\mu)\right)'}.$$

Thus, (3.19) holds.

This finishes the proof of Theorem 3.4.

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